Lecture 8

Quantum Chromo Dynamics

QCD is quantum field theory of strong interactions
describes couplings of quarks and gluons through colour

- Feynman Rules for QCD
- SU(3) Group Symmetry
- Colour States of Quarks & Gluons
- Strong Coupling Constant $\alpha_s$
- Asymptotic Freedom & Confinement
- QCD Potential
Rules for QCD Feynman diagrams

• A quark has one of three colour states (red, green or blue). An antiquark has one of three anticolours.

• A gluon propagator has a colour and an anticolour.
  – There are eight possible gluon states
  – Gluons have strong interactions with each other

• Colours are conserved at quark-gluon vertices
  – A quark-gluon vertex has a factor \(-ig_s\lambda^a\gamma^\mu\).
  – \(\lambda^a\) are generator matrices of SU(3) colour symmetry.
  – The quark and gluon colours are combined with the \(\lambda^a\) to give an overall colour factor, \(c_f\), in the amplitude.

• The coupling constant \(g_s = \sqrt{\alpha_s}\) is a function of \(q^2\):
  – At small \(q^2\) it is \(O(1)\), and QCD is non-perturbative
  – At large \(q^2\) it is smaller, and QCD becomes perturbative
Gluon Self Interactions

Gluons interact with each other through their colour and anticolour states. There can be three or four gluon vertices:

The three gluon vertex has a complicated factor:

\[-g_s f^{abc} \left[ g_{\mu\nu}(q_1 - q_2)\lambda + g_{\nu\lambda}(q_2 - q_3)\mu + g_{\lambda\mu}(q_3 - q_1)\nu \right] \]

The color structure constants $f^{abc}$ are related to the $\lambda^a$ matrices:

\[[\lambda^a, \lambda^b] = 2i \sum_c f^{abc} \lambda^c\]
SU(3) Colour Symmetry

An SU(3) group symmetry is related to colour conservation

• Three colour quantum numbers are separately conserved
• Strong interaction coupling $g_s$ is same for all colour states
• Invariance under rotation in three-dimensional colour space

$$U = e^{-i\alpha_a \cdot \lambda^a}$$

is a “non-Abelian” gauge symmetry

There are two types of “colourless” states:

Mesons have symmetric colour-anticolour wavefunctions:

$$\chi_c = \frac{1}{\sqrt{3}} (r\bar{r} + b\bar{b} + g\bar{g})$$

Baryons have antisymmetric three-colour wavefunctions:

$$\chi_c = \frac{1}{\sqrt{6}} (rgb - rbg + brg - bgr + gbr - grb)$$
The $\lambda$ matrices of SU(3)

\[
\begin{align*}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{align*}
\]
Colour states of Quarks and Gluons

Three quark states:
\[ r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

Gluon states are colour octet of SU(3) (from \( \lambda^a \)):

\[ G_1 = \frac{1}{\sqrt{2}} (r \bar{b} + \bar{r}b) \quad G_2 = \frac{-i}{\sqrt{2}} (r \bar{b} - \bar{r}b) \]
\[ G_4 = \frac{1}{\sqrt{2}} (r \bar{g} + \bar{r}g) \quad G_5 = \frac{-i}{\sqrt{2}} (r \bar{g} - \bar{r}g) \]
\[ G_6 = \frac{1}{\sqrt{2}} (b \bar{g} + \bar{b}g) \quad G_7 = \frac{-i}{\sqrt{2}} (b \bar{g} - \bar{g}b) \]
\[ G_3 = \frac{1}{\sqrt{2}} (r \bar{r} - b \bar{b}) \quad G_8 = \frac{1}{\sqrt{6}} (r \bar{r} + b \bar{b} - 2g \bar{g}) \]

No colour singlet gluon state: \( G_0 = \frac{1}{\sqrt{3}} (r \bar{r} + g \bar{g} + b \bar{b}) \)
Quark-Antiquark Scattering

Matrix element for quark-antiquark scattering:

\[-iM = \left[ \bar{u}_3 c_3^\dagger \right] \left[ \frac{-ig_s}{2} \lambda^a \gamma^\mu \right] \left[ u_1 c_1 \right] \left[ \frac{-ig_{\mu\nu} \delta^{ab}}{q^2} \right] \left[ \bar{v}_2 c_2^\dagger \right] \left[ \frac{-ig_s}{2} \lambda^b \gamma^\nu \right] \left[ v_4 c_4 \right] \]

\[\mathcal{M} = c_f \frac{\alpha_s}{4q^2} \left[ \bar{u}_3 \gamma^\mu u_1 \right] \left[ \bar{v}_2 \gamma_\mu v_4 \right]\]

where the colour factor is:

\[c_f = (c_3^\dagger \lambda^a c_1) (c_2^\dagger \lambda^a c_4)\]
**Colour factors in \( qq \) and \( q\bar{q} \) Scattering**

<table>
<thead>
<tr>
<th>quark states</th>
<th>gluon states</th>
<th>( c_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( rr \leftrightarrow rr )</td>
<td>( G_7, G_8 )</td>
<td>2/3</td>
</tr>
<tr>
<td>( r\bar{r} \leftrightarrow r\bar{r} )</td>
<td>( G_7, G_8 )</td>
<td>-2/3</td>
</tr>
<tr>
<td>( rb \leftrightarrow rb )</td>
<td>( G_7, G_8 )</td>
<td>-1/3</td>
</tr>
<tr>
<td>( rb \leftrightarrow br )</td>
<td>( G_1, G_2 )</td>
<td>1</td>
</tr>
<tr>
<td>( r\bar{r} \leftrightarrow b\bar{b} )</td>
<td>( G_1, G_2 )</td>
<td>-1</td>
</tr>
<tr>
<td>( r\bar{b} \leftrightarrow r\bar{b} )</td>
<td>( G_7, G_8 )</td>
<td>1/3</td>
</tr>
</tbody>
</table>

... and similarly for the other colour combinations by replacing \( r \to b, r \to g \) and \( b \to g \) and assigning the relevant gluon states.

See Halzen & Martin Pp.67-69 for detailed calculations of \( c_f \)
Strong Coupling Constant $\alpha_s$

Hard to measure running of $\alpha_s$ at low mass scale $\mu$!

$\alpha_s$ is measured to a few % at $\mu = M_Z$.
Description of Running of $\alpha_s$

In strong interactions the running of $\alpha_s$ is due to:

- Screening of colour by quark-antiquark ($f \bar{f}$) pairs
- *Anti-screening* of colour by gluons

Anti-screening of gluons dominates $\Rightarrow \alpha_s$ decreases with $q^2$

$$\alpha_s(q^2) = \frac{12\pi}{(33 - 2N_f) \ln \left( \frac{q^2}{\Lambda_{QCD}^2} \right)} \quad (\text{QCD})$$

$N_f \leq 6$ is the number of available quark flavours at a given $q^2$

$\Lambda_{QCD} = 217 \pm 25 \text{ MeV}$ is the QCD scale parameter

Reminder - in QED screening of electric charge by $f \bar{f}$ pairs:

$$\alpha(q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \ln \left( \frac{q^2}{\mu^2} \right)} \quad (\text{QED})$$
Azymptotic Freedom

At low $q^2 \approx \Lambda_{QCD}^2$ quarks and gluons are tightly bound inside meson and baryons

QCD calculations in this region are non-perturbative

Usually solved by numerical methods (Lattice QCD)

*Example: baryon and meson masses*

At high $q^2 \gg \Lambda_{QCD}^2$ quarks and gluons are *azymptotically free*

QCD calculations in this region are perturbative

Can be solved by summing diagrams in powers of $\alpha_s$ (like QED)

*Example: Gluon emission and hard scattering in DIS*

It is difficult to understand transition between two regions

*Example: fragmentation of partons into jets of hadrons*
Confinement

Quarks and gluons are confined inside hadrons and cannot be directly observed as free partons.

There are many models of confinement:

- The valence quark model of hadrons (next lecture)
  \( \text{Ignores effects of sea quarks and gluons at low } x \) 
- The colour flux-tube model
- The QCD potential model

None of these models provides a rigorous approach to non-perturbative QCD but they are conceptually useful.
Colour Flux-tube Model

Colour field lines compressed into flux-tube between quarks

Energy stored is like in an elastic string

Can break into $q\bar{q}$ pair:

Potential $V = kr \Rightarrow$ need infinite energy to separate quarks
The QCD potential $V_{q\bar{q}}(r)$ is given by:

$$V_{q\bar{q}}(r) = -\frac{4}{3} \frac{\alpha_s}{r} + kr$$

where $\alpha_s = 0.3$ and $k = 1$ GeV fm$^{-1}$. Mesons are $q\bar{q}$ bound states in a QCD potential well.