Lecture 6
Electron-Proton Scattering

- Mott Scattering Formula
- Elastic Scattering
- Electric & Magnetic Form Factors
- Deep Inelastic Scattering
- Structure Functions
Electrons (and muons) are used to probe the sub-structure of protons (and neutrons)

\[ e^- (p_1) \rightarrow q \rightarrow i e \gamma^\mu \rightarrow e^- (p_3) \]

\[ p(p_2) \rightarrow -i e K^\mu \rightarrow p(p_4) \]

Scattering is off quarks by **electromagnetic interactions**
Mott Scattering

Scattering of a relativistic electron by a pointlike spin 1/2 proton

Similar to electron muon scattering from last Lecture

Usually described in the Lab frame, where the proton is at rest:
\( \theta \) is the lab scattering angle of the electron
\( p_e \) is the incident electron beam momentum
\( q^2 \) is the four-momentum transfer of the virtual photon

Mott scattering formula:

\[
\frac{d\sigma}{d\Omega}\bigg|_{\text{point}} = \frac{\alpha^2}{4p_e^2 \sin^4 \frac{\theta}{2}} \left( \cos^2 \frac{\theta}{2} - \frac{q^2}{2m_p^2} \sin^2 \frac{\theta}{2} \right)
\]

This is a modified version of Rutherford scattering
The second term in brackets accounts for the proton recoil
Elastic Scattering Kinematics

Energy lost by electron due to proton recoil (“inelasticity”):

\[ \nu = E_1 - E_3 \quad \nu > 0 \]

In elastic scattering there is a simple relationship between \( \nu \) and \( q^2 \) from four-momentum conservation:

\[ 2m_p \nu = Q^2 = -q^2 \quad q^2 < 0 \quad Q^2 > 0 \]

*The four-momentum squared of the virtual photon is negative!*

Inelastic electron-proton scattering is described by the two variables \( \nu \) and \( q^2 \) (or \( Q^2 \))

Be careful not to get confused between \( q^2 \) and \( Q^2 \)

Most textbooks and these lectures use both!
Form Factors

Deviations from the Mott scattering formula describe the charge distribution inside the proton in terms of a form factor $F(q^2)$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{\text{point}} |F(q^2)|^2$$

Low $q^2$ probes distances larger than size of proton ($r \approx 1\text{fm}$)

There is no sensitivity to charge distribution $F(0) = 1$

Large $q^2$ probes inside the proton and the form factor $F(q^2) < 1$

Form factor is Fourier transform of charge distribution:

$$F(q^2) = \int \rho(\vec{x}) e^{i\vec{q} \cdot \vec{x}} d^3x$$
Mean square charge radius of proton:

\[ \rho(r) = \rho_0 e^{-r/r_0} \quad <r_0^2> = 0.8\text{fm} \]
Proton Electromagnetic Current

Matrix element for elastic scattering

\[ \mathcal{M}(e^- p \rightarrow e^- p) = \frac{e^2}{(p_1 - p_3)^2} (\bar{u}_3 \gamma^\mu u_1) (\bar{u}_4 K_\mu u_2) = \frac{1}{q^2} j_{e\mu} j_{p\mu} \]

can be factorized into lepton and proton electromagnetic currents

Proton current can be written in terms of a “Dirac charge” form factor \( F_1 \) and an “anomalous magnetic” form factor \( F_2 \):

\[ j_\mu^p = e\bar{u}_4 \left( \gamma^\mu F_1(q^2) + i\kappa_p \frac{F_2(q^2)}{2m_p} \sigma^{\mu\nu} q_\nu \right) u_2 \]

where \( \kappa_p \) is the anomalous magnetic moment of the proton

\[ \mu_p = \frac{(1 + \kappa_p)e}{2m_p} \quad \kappa_p = 1.79 \]
Differential Cross-section

Differential cross-section for elastic scattering is written in terms of the form factors $F_1$ and $F_2$:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2} E_1} \left\{ \left( F_1^2 - \frac{\kappa_p q^2}{4m_p^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2m_p^2} (F_1 + \kappa_p F_2)^2 \sin^2 \frac{\theta}{2} \right\}$$

Low $q^2$ limit has $F_1^p(0) = 1, F_2^p(0) = 1$

A pointlike proton would have $F_1(q^2) = 1$ for all $q^2$ and $\kappa_p = 0$

(This is Mott scattering)

Even though the neutron has no charge it does have an anomalous magnetic moment $\kappa_n = -1.91, \mu_n = \kappa_n e / 2m_n$

In the low $q^2$ limit $F_1^n(0) = 0, F_2^n(0) = 1$
Electric and Magnetic Form Factors

Define linear combinations of $F_1$ and $F_2$:

$$G_E = F_1 + \frac{\kappa q^2}{4m_p^2} F_2 \quad G_M = F_1 + \kappa F_2$$

Differential cross-section becomes:

$$\left. \frac{d\sigma}{d\Omega} \right|_{\text{lab}} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \frac{E_3}{E_1} \left( \frac{G_E^2 + \tau G_M^2}{1 + \tau} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)$$

where $\tau = Q^2/4m_p^2$ and $G_M$ is associated with the proton recoil

$G_E$ and $G_M$ are known as electric and magnetic form factors

The ratios of the proton form factors are constrained to be:

$$G_E = \frac{G_M}{\mu_p} \quad F_1 = \kappa_p \frac{(1 + \mu_p \tau)}{\mu_p - 1} F_2$$
Deep Inelastic Scattering

During deep inelastic scattering (DIS) at high $q^2$ the proton breaks up into its constituent quarks:

The quarks form a hadronic jet with invariant mass $W$
Kinematics of DIS

The invariant mass squared of the hadronic jet is:

$$W^2 = m_p^2 + 2m_p \nu + q^2$$

Since $W \neq m_p$, the four momentum transfer $q^2$ and inelasticity $\nu$ are independent variables.

They are usually replaced by the parton energy $x$:

$$x = \frac{Q^2}{2m_p\nu} = \frac{-q^2}{2m_p\nu}$$

and the parton rapidity $y$:

$$y = \frac{p_2 \cdot q}{p_2 \cdot p_1} = \frac{\nu}{E_1}$$

$x, y$ are dimensionless variables with ranges $0 \leq (x, y) \leq 1$
Matrix element for DIS

The matrix element squared can be factorised into lepton and hadronic current terms:

$$|\mathcal{M}|^2 = \frac{e^4}{q^2} L_\mu^\nu (W_{\text{hadron}})_\mu^\nu$$

The hadronic part describes the inelastic breakup of the proton in terms of two structure functions $W_1$ and $W_2$ which depend on the kinematic variables $\nu$ and $Q^2$ (or equivalently on $x$ and $y$)

The doubly differential cross-section is:

$$\frac{d\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \left( W_2(\nu, Q^2) \cos^2 \frac{\theta}{2} + 2W_1(\nu, Q^2) \sin^2 \frac{\theta}{2} \right)$$
Measurement of DIS cross-section

First done at SLAC in 1970s
(Friedman, Kendall & Taylor: Nobel Prize 1990)

Peaks are from proton (elastic) and baryonic resonances
**Structure Functions $F_1$ and $F_2$**

In Deep Inelastic Scattering it is usual to replace the structure functions $W_1, W_2$ with $F_1, F_2$:

$$m_p W_1(\nu, Q^2) \rightarrow F_1(x) \quad \nu W_2(\nu, Q^2) \rightarrow F_2(x)$$

**Warning** - $F_{1,2}(x)$ in DIS are **not** the same as $F_{1,2}(q^2)$ in elastic scattering!

Note that $F_1$ and $F_2$ are only written as functions of $x$!

$x$ is fraction of the proton energy off which the scattering occurs:

$$x = \frac{Q^2}{2m_p \nu} = \frac{m}{m_p} \quad 2m\nu + q^2 = 0$$

Implies that the virtual photon interacts with a point-like spin 1/2 **parton** inside the proton (i.e. a quark)
Measurements of Structure functions

Are data for a given $x$ independent of $Q^2$?

Some dependence on $Q^2$
(more next lecture)