Lecture 3

Experimental Methods & Feynman Diagrams

- Natural Units & the Planck Scale
- Review of Relativistic Kinematics
- Cross-Sections, Matrix Elements & Phase Space
- Decay Rates, Lifetimes & Branching Fractions
- Feynman Diagrams
- Particle Wavefunctions
Natural Units

At the Planck scale the natural units have $G = 1$, $\hbar = 1$ and $c = 1$

Planck Mass or Energy: $M_{Pl} = \sqrt{\frac{\hbar c}{G}} = 1.2 \times 10^{19}$ GeV/c$^2$

Planck length: $l_{Pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35}$ m

Planck time: $t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} = 5.4 \times 10^{-44}$ s

I will use $\hbar = 1$, $c = 1$. To recover SI units you need to use dimensional analysis to introduce the correct factors of $\hbar$ and $c$.

It is useful to remember the approximate values:
$\hbar = 1 \times 10^{-34}$ Js, $c = 3 \times 10^8$ m/s, $\hbar c = 197$ MeV fm

A reminder that 1eV = $1.6 \times 10^{-19}$ J and 1fm = $10^{-15}$ m
Relativistic Kinematics

Four momentum components are always conserved

Lorentz transformation:

\[ p^\mu = [E, \vec{p}] \quad p'_x = \gamma(p_x - \beta E) \quad E' = \gamma(E - \beta p_x) \]

where \( \beta = p/E \) and \( \gamma = 1/\sqrt{1 - \beta^2} = E/m \)

Products of four-vectors are Lorentz invariants

Invariant mass

\[ p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2 \quad E^2 = |\vec{p}|^2 + m^2 \]

It is often possible to take one of these limits:

Highly relativistic particle: \( m \ll E, E \approx p, \beta \to 1, v \to c, \gamma \to \infty \)

Non-relativistic particle: \( p \ll m, E = mc^2 + \frac{1}{2}mv^2, \beta \to 0, \gamma \to 1 \)
Two-body Collisions

CM frame: $\vec{p}_1 = -\vec{p}_2$, $\vec{p}_3 = -\vec{p}_4$, $|\vec{p}_1| = |\vec{p}_2| = p_i$, $|\vec{p}_3| = |\vec{p}_4| = p_f$

For elastic collisions $m_1 = m_3$, $m_2 = m_4$, $p_i = p_f$

Lorentz Invariant ("mass squared")

$$s = (E_1 + E_2)^2 = (E_3 + E_4)^2$$

Lab frame: $|\vec{p}_2| = 0$, $\vec{p}_1 = \vec{p}_{\text{beam}}$, $\vec{p}_3 = \vec{p}_{\text{scatter}}$, $\vec{p}_4 = \vec{p}_{\text{recoil}}$

$$s = (p_1 + p_2)^2 = (E_1 + m_2)^2 - |\vec{p}_1|^2 = m_1^2 + m_2^2 + 2E_{\text{beam}}m_2$$

Lorentz transformation from lab to CM frame:

$$\beta = \frac{\vec{p}_{\text{beam}}}{(E_{\text{beam}} + m_2)} \quad \gamma = \frac{(E_{\text{beam}} + m_2)}{\sqrt{s}}$$
Cross-sections & Phase Space

Cross-section $\sigma$ has dimensions $m^2$ (1 barn = $10^{-28}m^2$). Transition rate between initial and final states $W_{fi}$ is normalized to incident flux, where $v_i$ is the relative velocity of the initial state.

$$\sigma = \frac{W_{fi}}{N_i v_i} = \frac{2\pi}{\hbar} |M_{fi}|^2 \frac{\rho_f}{N_i v_i}$$

The physics of the underlying interaction is contained in the matrix element, $M_{fi}$.

$\rho_f$ is the energy density $dN_f/dE$ of the final states, known as the phase space. For a two-body collision in the CM frame:

$$\rho_f = \frac{1}{(2\pi\hbar)^3} p_f^2 \frac{dp_f}{dE_0} d\Omega = \frac{1}{(2\pi\hbar)^3} \frac{p_f^2}{v_f} d\Omega$$

$E_0 = \sqrt{s}$ and $v_f$ is the relative velocity of the final state particles. Note that $\sigma$, $|M_{fi}|$, $v_i$ and $\rho_f$ are all Lorentz invariants.
**Differential and Polarized Cross-sections**

Can measure angular distributions $d\sigma/d\Omega$, where $d\Omega = d(\cos \theta)d\phi$

In two-body collisions angular dependence comes from $M_{fi}(\theta, \phi)$

Can measure cross-sections for two-body fermion collisions with specific initial and final state polarizations, e.g. $\sigma(\uparrow\downarrow\uparrow\downarrow)$ may be different from $\sigma(\uparrow\downarrow\downarrow\uparrow\uparrow)$.

Again the spin dependence comes from $M_{fi}(\uparrow\downarrow\uparrow\downarrow)$ and $M_{fi}(\uparrow\downarrow\downarrow\uparrow\uparrow)$

A lot of information about the matrix element of an interaction comes from angular and polarization measurements

To calculate $M_{fi}$ for unpolarized cross-sections:

- **initial** state spins are **averaged** over

- **final** state spins are **summed** over
Decay Rates & Lifetimes

Partial decay width (in GeV) of a particle to a final state $f$:

$$
\Gamma_f = \hbar W_{fi} = 2\pi |M_{fi}|^2 \rho_f
$$

Final state phase space is constrained by four-momentum conservation. Two-body decay of a particle of mass $m_i$ at rest has:

$$
\rho_f = \frac{1}{(4\pi)^2} \frac{p_f}{m_i^2}
$$

There can be several different decay modes. Total decay width and partial branching fraction:

$$
\Gamma = \sum_f \Gamma_f \quad B_f = \frac{\Gamma_f}{\Gamma}
$$

Proper lifetime (in rest frame), and decay length of moving particle

$$
\tau = \frac{\hbar}{\Gamma} \quad L = \gamma\beta c\tau
$$
Feynman Diagrams

A *pictorial* representation of fermion and boson interactions designed to help with matrix element calculations.

Example of electron-electron (Moeller) scattering:

\[ e^- \longrightarrow \gamma \rightarrow e^- \]

\[ \gamma \longrightarrow e^- \rightarrow e^- \]

*N.B. Initial state is on the left, and final state on the right, but the horizontal and vertical axes are not strictly related to space or time!"
Drawing Feynman Diagrams

- Initial state particles enter from the left.
- Final state particles exit to the right.
- A line between two vertices is a “virtual particle” (*virtual particles cannot be observed!*)
- Fermions are solid lines with arrows pointing to right.
- Antifermions have arrows pointing to left.
- Photons are represented by wavy lines.
- Gluons are represented by springs.
- Heavy bosons \((W, Z, H)\) are dashed lines.
Four momentum conservation

Each particle has a four momentum $p_\mu = [E, \vec{p}]$

Need to define these in a **frame of reference**

- Initial and final state particles have $E^2 - |\vec{p}|^2 = m^2$
  where $m$ is the rest mass of the fermion or boson

- Virtual particles have $E^2 - |\vec{p}|^2 = q^2$
  where $q$ is not the rest mass of a physical particle.
  A virtual particle is said to be “off the mass shell”.
  Note that $q^2$ can be positive or negative!

- Four momentum is conserved **at each vertex**

- Four momentum is conserved between initial and final states
Rules for writing down amplitudes

- Initial and final state particles have wavefunctions
  - Spin 0 bosons ⇒ plane waves
  - Spin 1/2 fermions ⇒ spinors
  - Spin 1 bosons ⇒ polarizations

- Vertices have dimensionless coupling constants
  - Electromagnetic interactions ⇒ $\sqrt{\alpha} \propto e$
  - Strong interactions ⇒ $\sqrt{\alpha_s}$

- Virtual particles have propagators
  - Virtual photon has propagator $\propto 1/q^2$
  - Virtual boson of mass $m$ has propagator $\propto 1/(q^2 - m^2)$

The sum of the amplitudes of all possible Feynman diagrams gives the Matrix element for the interaction
Particle Wavefunctions

Free particle wavefunction in four-vector notation:

$$\psi = N e^{-ip^\mu x_\mu}$$

*This contains no charge, spin, colour or flavour information!*

Normalisation is determined from the density of states in a box:

$$\rho = i \left[ \psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi \right]$$

$$\int_V \rho d^3x = 2E \quad N = \frac{1}{\sqrt{V}}$$

When calculating physical cross-sections including phase space, the volume of the box $V$ drops out and can be taken to be one.

Probability current associated with particle transition $p_1 \rightarrow p_3$:

$$j_\mu = i [\psi^* (\partial_\mu \psi) - (\partial_\mu \psi^*) \psi] = (p_1 + p_3) e^{i(p_3 - p_1)x}$$
A spinless interaction

Hypothetical interaction where two spinless particles exchange a massless boson with coupling strength $g$

Particle currents $(p_i + p_f) e^{i(p_f - p_i)x}$

Propagator gives $1/q^2$

Vertex couplings $g$

Matrix element is:

$$\mathcal{M} = g^2 \frac{(p_1 + p_3)(p_2 + p_4)}{(p_4 - p_2)^2} \delta^4(p_3 + p_4 - p_1 - p_2)$$

where the $\delta$ function accounts for four-momentum conservation.
Mandelstam variables

Used for the momentum transfer $q^2$ in two-body collisions

Annihilation/creation of pairs of particles is s-channel

\[ s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 4p^2 \]

Scattering of particles is t or u-channel

\[ t = (p_1 - p_3)^2 = (p_4 - p_2)^2 = 2p^2(1 - \cos \theta) \]
\[ u = (p_1 - p_4)^2 = (p_3 - p_2)^2 = 2p^2(1 + \cos \theta) \]

where $p$ is CM momentum, and $\theta$ is CM scattering angle

Initial state flux factor and final state phase space:

\[ \frac{1}{v_i} = \frac{1}{p\sqrt{s}} \quad \rho_f \propto \frac{p}{\sqrt{s}} \quad \frac{\rho_f}{v_i} \propto \frac{1}{s} \]
Identical particles

The Mandelstam variables can be used to express the spinless cross-section more compactly:

\[ \mathcal{M} = g^2 \frac{(u - s)}{t} \]

If the spinless particles are identical then there are two diagrams:

\[ p_1 \to p_3, \ p_2 \to p_4 \quad \text{OR} \quad p_1 \to p_4, \ p_2 \to p_3 \]

The matrix element becomes:

\[ \mathcal{M} = g^2 \left[ \frac{(u - s)}{t} + \frac{(t - s)}{u} \right] \]

The two diagrams are related by an exchange symmetry \( t \leftrightarrow u \)