GENERALIZED SEMIEMPIRICAL EQUATIONS FOR THE EXTRAPOLATED RANGE OF ELECTRONS

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The extrapolated or practical range \( R_{\text{ex}} \) of monoenergetic electrons in the energy region 0.3 keV-30 MeV for the absorbers of atomic number 6-92 has been found to be expressed by a single semiempirical equation of the form

\[
R_{\text{ex}} = a_1 \left\{ \frac{1}{a_2} \ln\left(1 + a_2 \tau\right) - \frac{a_2}{1 + a_2 \tau^2} \right\},
\]

where \( \tau \) is the incident kinetic energy in units of the rest energy of the electron, and the parameters \( a_i \) (\( i = 1, 2, \ldots, 5 \)) are given by simple functions of atomic number \( Z \). Values of nine constants to express \( a_i \) have been determined by least-squares fit to a total of 232 experimental points reported in 18 references; the data used have been confined to those obtained from number-transmission measurement. The rms deviation of the equation from the experimental data is 4.5% for energies above 1 MeV and 8.4% for the entire energy region. Although the latter value is rather large, it is mainly due to large fluctuation of the experimental data for energies below 1 MeV. An approximate inverse relation which expresses \( \tau \) as a function of \( R_{\text{ex}} \) is also given.

1. Introduction

The extrapolated or practical range \( R_{\text{ex}} \) of electrons is useful for various purposes in measurement and utilization of electrons, for example, design of detection systems, simple energy determination, and estimation of effects associated with deep penetration of electrons. It is usually defined as the thickness of material at which the extension of the linearly decreasing region of the transmission curve (number-thickness curve) becomes zero. At low energies, \( R_{\text{ex}} \) is frequently determined from linear extrapolation of the number-energy curve measured for a given thickness of the absorber, and the two definitions have been found to yield consistent results. For absorbers of high atomic number, the transmission curve often does not show the linear region, and the extrapolation is then made of the tangent at the steepest point of the curve.*

A number of authors have given empirical equations to express the relation between \( R_{\text{ex}} \) and the incident kinetic energy \( T \) of the electron. Earlier works on this problem were mainly restricted to the case of Al absorbers, and have been reviewed by Katz and Penfold. Later works are briefly reviewed in the appendix of the present paper. Although formulation of the dependence of \( R_{\text{ex}} \) on atomic number \( Z \) and atomic weight \( A \) of the absorber has been attempted, the results are applicable only in limited energy regions.

The purpose of the present paper is to describe a semiempirical expression for \( R_{\text{ex}}(T; Z, A) \) and an expression for \( T(R_{\text{ex}}; Z, A) \), the latter being an approximate inverse function of the former. Each of these expressions, given in a single form, covers wider regions of \( T \) and \( Z \) than any of previous equations.

2. Formulation

In order to derive the functional form of a generalized equation for \( R_{\text{ex}} \), we first consider an expression for the continuous slowing down approximation range \( R_0 \). It is defined by

\[
R_0 = \int_0^\tau S_{\text{tot}}^{-1} \, dT, \tag{1}
\]

where \( S_{\text{tot}} \) is given by

\[
S_{\text{tot}} = S_{\text{col}} + S_{\text{rad}}; \tag{2}
\]

\( S_{\text{col}} \) is the mass stopping power due to collision, and \( S_{\text{rad}} \) is the one due to radiation. Weber has shown that \( S_{\text{col}} \) can be well approximated by

\[
S_{\text{col}} = \left\{ C_1 [1 - C_2 (1 + C_3 T)^{-2}] \right\}^{-1}, \tag{3}
\]

where \( C_i \) (\( i = 1, 2, 3 \)) is a constant for a given absorber. On the other hand, the ratio \( S_{\text{rad}}/S_{\text{col}} \) is given approximately by

\[
S_{\text{rad}}/S_{\text{col}} \approx TZ/1600mc^2 \equiv \alpha T, \tag{4}
\]

* Because of ambiguities accompanying the extrapolation in such a case, Dupouy et al. have proposed to use the characteristic thickness \( R_c \) instead of \( R_{\text{ex}} \); it is defined as the thickness at which the transmission coefficient has fallen to 5%. While \( R_c \) is numerically almost the same as \( R_{\text{ex}} \) in the case of the Al absorber, the difference is appreciable for absorbers of higher atomic number. The present authors consider that each of \( R_{\text{ex}} \) and \( R_c \) is useful in different context.
where \( mc^2 \) is the rest energy of the electron. Substituting eqs. (2)-(4) into eq. (1), we obtain the relation

\[
R_0 \approx C_1 [(1/\alpha) \ln (1 + \alpha T) - C_2 T/(1 + C_3 T)],
\]

where the terms of order \( \alpha/C_3 (\approx 10^{-3}Z) \) have been neglected compared with those of order unity*. The relation between \( R_{ex} \) and \( T \) is expected to be fitted with a function similar to the right hand side of eq. (5), though the behavior of \( R_{ex} \) and \( R_0 \) differs slightly owing to the influences of multiple scattering detours and energy-loss straggling exerted upon the former. Actually a good fit to the existing experimental data (the sources of the data used are described in section 4) has been obtained by introducing a single additional parameter. The fitting function can be written as

\[
R_{ex} = a_1 [(1/a_2) \ln (1 + a_2 \tau) - a_3 \tau/(1 + a_4 \tau^2)],
\]

where \( a_i (i=1,2,\ldots,5) \) is a constant for a given absorber and \( \tau = T/mc^2 \). When \( a_2 \tau \ll 1 \) and \( a_3 = 1 \), the relation (6) reduces to Weber’s equation [eq. (A3) in the appendix].

For the purpose of finding the dependences of \( a_i \) on \( Z \) and \( A \), the values of \( a_i \) for representative elements have been determined by least-squares fit to the experimental data. From an analysis of the results, the following relations have been found:

\[
\begin{align*}
a_1 &= b_1 A/Z^{b_2}, \\
a_2 &= b_3 Z, \\
a_3 &= b_4 - b_5 Z, \\
a_4 &= b_6 - b_7 Z, \\
a_5 &= b_8 /Z^{b_9},
\end{align*}
\]

where the symbols \( b_i \) denote constants independent of absorber material. In eq. (6a), the constant \( b_2 \) is close to unity, and the approximate proportionality of \( a_2 \) to \( A/Z \) is reasonable considering the known behavior of the corresponding parameter \( C_1 \) in eq. (5). The form of eq. (6b) for \( a_2 \) is also expected from the dependence of \( \alpha \) on \( Z \) [see eq. (4)].

Since it is difficult to solve eq. (6) for \( \tau \), we formulate an approximate inverse relation for convenience. It is of the following form:

\[
\tau = c_1 \left( \exp \{ R_{ex} c_2 + c_3 (1 + c_4 R_{ex}^2) /c_1 \} - 1 \right),
\]

where \( c_i (i=1,2,\ldots,5) \) is a constant for a given absorber. The functional form of eq. (7) can be obtained by replacing the variables \( \tau \) in the second term between brackets of eq. (6) with \( R_{ex} \) multiplied by a constant and by solving for the remaining \( \tau \). The dependences of \( c_i \) on \( Z \) have been determined by the same method as used in finding eqs. (6a)-(6e), and are given by

\[
\begin{align*}
c_1 &= d_1 /Z, \\
c_2 &= d_2 Z^{d_3} /A, \\
c_3 &= d_4 - d_5 Z, \\
c_4 &= d_6 /Z^{d_7}, \\
c_5 &= d_8 /Z^{d_9},
\end{align*}
\]

where the symbols \( d_i \) denote constants independent of absorber material. The forms of eqs. (7a) and (7b) are expected from the fact that the parameters \( c_1 \) and \( c_2 \) are approximately the reciprocals of \( c_2 \) and \( c_1 \), respectively.

Hereafter, the expression (6) in which eqs. (6a)-(6e) are substituted are simply called eq. (6), and similar implication is carried by eq. (7).

3. Determination of the constants

Values of the constants \( b_i \) and \( d_i \) have been sought by the least-squares fit with eqs. (6) and (7) to the experimental data, respectively. Since the result of a preliminary fit has shown that the data for \( Z = 4 \) cannot be well fitted with eq. (6), the final computation has been made by using a total of 232 data points for \( Z \geq 6 \).

Uncertainties of the experimental values are not shown in all the sources of the data used. Therefore, each data point has been given equal significance in the sense that \( \sigma_i /R_{ex} \), where \( R_{ex} \) is the \( i \)th experimental value of \( R_{ex} \), and \( \sigma_i \) is its uncertainty or the standard deviation about the “true” value. Strictly speaking, this assumption is not always correct, but is the most simple and convenient for obtaining the best fit to the measurements of the quantity whose value varies over more than a few orders of magnitude.

4. Sources of the data

The sources of the experimental data used are presented in table 1. Only the data obtained from number-transmission measurement have been employed, and those obtained from ionization-in-depth measurement have been excluded, because it has been pointed out\(^{24,26} \) that the latter technique leads to slightly different results. Although the previous
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TABLE 1

Sources of the data used to determine the values of the constants in the empirical equations. The energy region and the absorber atomic numbers of each experiment are also shown. The data for Z = 4 have been excluded in the final computation of least-squares fit.

<table>
<thead>
<tr>
<th>Author</th>
<th>T (MeV)</th>
<th>Z</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kanter, Sternglass a</td>
<td>0.00076–0.00498</td>
<td>6, 13, 28, 47, 79</td>
<td>18</td>
</tr>
<tr>
<td>Kanacheva, Burtsev b</td>
<td>0.0014–0.0049</td>
<td>79</td>
<td>4</td>
</tr>
<tr>
<td>Cosslett, Thomas</td>
<td>0.0025–0.015</td>
<td>13, 29, 47, 79</td>
<td>5</td>
</tr>
<tr>
<td>Schönland</td>
<td>0.0102–0.0876</td>
<td>13</td>
<td>19</td>
</tr>
<tr>
<td>Schönland</td>
<td>0.0185–0.0285</td>
<td>13, 29, 47, 79</td>
<td>20</td>
</tr>
<tr>
<td>Lane, Zaffarano</td>
<td>0.0273</td>
<td>13</td>
<td>21</td>
</tr>
<tr>
<td>Eddy</td>
<td>0.1465–0.576</td>
<td>13</td>
<td>22</td>
</tr>
<tr>
<td>Seliger b</td>
<td>0.159–0.96</td>
<td>13, 47, 50, 79, 82</td>
<td>23</td>
</tr>
<tr>
<td>Agu et al. c</td>
<td>0.25–0.75</td>
<td>4, 13, 29, 47, 79</td>
<td>24</td>
</tr>
<tr>
<td>Miller, Hendricks</td>
<td>0.25–1</td>
<td>13</td>
<td>25</td>
</tr>
<tr>
<td>Grumshaw</td>
<td>0.36–1.86</td>
<td>13</td>
<td>26</td>
</tr>
<tr>
<td>Nakai et al.</td>
<td>1–1.8</td>
<td>4, 13, 47</td>
<td>27</td>
</tr>
<tr>
<td>Nakai et al. b</td>
<td>1–2</td>
<td>13, 29, 47</td>
<td>28</td>
</tr>
<tr>
<td>Miller</td>
<td>1–3</td>
<td>13, 22, 23, 29, 79</td>
<td>29</td>
</tr>
<tr>
<td>Nakai et al.</td>
<td>1, 5, 2.1</td>
<td>6</td>
<td>30</td>
</tr>
<tr>
<td>Ebert et al. b</td>
<td>4–12</td>
<td>6, 13, 29, 47, 73, 92</td>
<td>13</td>
</tr>
<tr>
<td>Tabata et al.</td>
<td>4.09–23.5</td>
<td>4, 13, 29, 47, 79</td>
<td>14</td>
</tr>
<tr>
<td>Harder, Poschet</td>
<td>4.17–30</td>
<td>6, 13, 29, 48, 82</td>
<td>31</td>
</tr>
</tbody>
</table>

a Numerical data obtained through private communication from the authors have been used.
b Values have been read from the graphs in the paper.
c Values of $R_{ex}/R_0$ have been read from the graph except in the case of $Z = 13$, for which numerical data are given, and have been multiplied by the values of $R_0$ given in refs. 32, 33.

Authors have also used the data for Al$_2$O$_3$, Al-coated Formvar, and Al-coated collodion in determining the empirical constants for Al, the data for such composite absorbers have not been used in the present treatment. The data of Viatskin and Makarov have not been adopted because of rather large systematic deviations from the other data.

Since the data of Tabata et al. for the effectively semiinfinite medium show no significant difference from the data for the usual finite-slab geometry, they have been included.

5. Results and discussion

Values of the constants $b_i$ and $d_i$ determined are listed in table 2. The relations given by eq. (6) with these values of $b_i$ are compared with the experimental data in figs. 1 and 2. In fig. 1, experimental data for Al$_2$O$_3$ (refs. 18, 35) are also plotted at the lowest energies, where the data for pure Al are absent. It can be seen from these figures that eq. (6) well reproduces the experimental data presently available in the energy region 0.3 keV–30 MeV. A measure of goodness of fit may be given by the relative rms deviation $\delta_{rms}$ of the empirical equation from the experimental data. It is defined by

$$
\delta_{rms} = \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{R_{ex}(\tau_i)}{(R_{ex})_{i-1}} - 1 \right]^2 \right\}^{\frac{1}{2}}, \quad \text{for eq. (6), (8)}
$$

$$
\delta_{rms} = \left[ \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{\tau_i}{(R_{ex})_{i-1}} - 1 \right]^2 \right]^{\frac{1}{2}}, \quad \text{for eq. (7), (9)}
$$

where $n$ is the number of experimental data, $\tau_i$ is the $i$th experimental value of $\tau$, and $R_{ex}(\tau_i)$ and $\tau_i[(R_{ex})_{i-1}]$ are the values of $R_{ex}$ and $\tau$ calculated from eqs. (6) and (7), respectively. Values of $\delta_{rms}$ are shown in table 3. When the entire energy region is considered, $\delta_{rms}$
is rather large. However, this is mainly due to large fluctuation of experimental points for $T<1$ MeV as can be seen from figs. 1 and 2, and $\delta_{\text{rms}}$ becomes considerably small when only the data for $T \geq 1$ MeV are taken into account (see the parenthesized values in table 3). Although there is also a trend that $\delta_{\text{rms}}$ is smaller for eq. (7) than for eq. (6), this can be attributed to the superficial effect of the fact that the logarithmic gradient $g = (\tau/R_{\text{ex}})dR_{\text{ex}}/d\tau$ of the $\tau-R_{\text{ex}}$ curve is greater than unity in larger part of the energy region treated here (the relative deviation in $R_{\text{ex}}$ is equal to the corresponding relative deviation in $\tau$ multiplied by $g$).

The energies calculated from eqs. (6) and (7) for a given value of $R_{\text{ex}}$ agree with each other within a few % in most cases. Although the maximum relative deviation $\delta_{\text{max}}$ of eq. (7) from eq. (6) exceeds 10% in the cases of the lowest and the highest values of $Z$ as shown in table 4, it is considered to be compatible with the uncertainty in the interpolation or the extrapolation of the presently available data.

The relations (6) and (7) can be applied also to the

Fig. 1. Comparison of the semiempirical relation (6) (lines) with experimental data (solid circle: Al, open circle: Cu). Use the scales as indicated. The three experimental points at the lowest energies are for Al$_2$O$_3$. 
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Fig. 2. Comparison of the semiempirical relation (6) (lines) with experimental data (solid circle: Ag, open circle: Au and Pb). Use the scales as indicated.

Absorbers which are mixtures or compounds. In this case, Z and A should be replaced by

\[ Z_{\text{eff}} = \sum f_i Z_i, \]

\[ A_{\text{eff}} = Z_{\text{eff}} (Z/A)_{\text{eff}}^{-1}, \]

respectively, where

\[ (Z/A)_{\text{eff}} = \sum_i f_i Z_i / A_i, \]

and \( f_i \) is the fraction by weight of the constituent element with atomic number \( Z_i \) and atomic weight \( A_i \).

Appendix

In this appendix, previous works concerned with equations for the extrapolated range \( R_{\text{ex}} \) and the characteristic thickness \( R_c \) are reviewed (the definition of \( R_c \) is given in the footnote on the first page of this paper).

Equation for \( R_{\text{ex}} \). Earlier works have been well summarized by Katz and Penfold\(^2\), so that we do not repeat them here. The equations proposed by them for
the Al absorber have been the most familiar, and are expressed in the form

\[ R_{ex} = B_1 T^{B_2 - B_3 \ln T}, \quad \text{for } 0.01 \leq T \leq 2.5 \text{ MeV}, \quad (A1) \]

\[ R_{ex} = B_1' T - B_2', \quad \text{for } T \geq 2.5 \text{ MeV}, \quad (A2) \]

where \( T \) is the incident kinetic energy of the electron, and \( B_i \) (\( i = 1, 2, 3 \)) and \( B_i' \) (\( i = 1, 2 \)) are constants.

Young\(^3\) has fitted the power law to his experimental data for Al in the energy region 0.5–11 keV. Kanicheva and Burtsev\(^4\) and Cosslett and Thomas\(^5\) have found that the power law is applicable also to metals other than Al for 2 \( \leq T \leq 15 \) keV.

Using the published experimental data for 0.2 \( \leq T \leq 3.5 \) MeV, Varshni and Karnatak\(^6\) have reevaluated the constants in the Flammersfeld equation\(^7\) for Al. Kahre and Varshni\(^8\) have proposed a modified Flammersfeld equation, extending the region of validity down to 2 keV.

Weber\(^9\) has semiempirically established an equation valid for Al in the energy region 3 keV–3 MeV. It is of the form

\[ R_{ex} = C_i T [1 - C_{ij} (1 + C_3 T)], \quad (A3) \]

where \( C_i \) (\( i = 1, 2, 3 \)) is a constant. Kobetich and Katz\(^{10}\) have made this equation applicable to the energy interval 0.3 keV–20 MeV by small adjustment of the constants.

For light materials and 3 \( \leq T \leq 15 \) MeV, Markus\(^{11}\) has proposed an equation of the same form as eq. (A2) with \( B_1' \) and \( B_2' \) proportional to \( (Z/A)_{\text{eff}}^{-1} \), where \( (Z/A)_{\text{eff}} \) is given by eq. (12) in section 5.

Expressing \( B_1 \) by a power function of atomic number \( Z \), Rao\(^{12}\) has made eq. (A1) applicable to the regions 100 keV \( \leq T \leq 1 \) MeV and 13 \( \leq Z \leq 82 \). Ebert et al.\(^{13}\) have expressed their experimental data for 4 \( \leq T \leq 12 \) MeV and 6 \( \leq Z \leq 92 \) with an equation of the same form as eq. (A2) in which \( 1/B_1' \) and \( 1/B_2' \) are linear functions of \( Z \). Tabata et al.\(^{14}\) have fitted their experimental results for 4 \( \leq T \leq 24 \) MeV and 4 \( \leq Z \leq 79 \) with an equation containing six parameters.

Equation for \( R_e \). Dupouy et al.\(^1\) have formulated an empirical equation to express the transmission curves observed for 50 keV \( \leq T \leq 1.2 \) MeV and 13 \( \leq Z \leq 79 \), and have derived from it an expression for \( R_e \) as a function of \( T, Z \), the atomic mass, and the density of absorber material. Kobetich and Katz\(^{15}\) have fitted the values of \( R_e \) given by the equation of Dupouy et al., with eq. (A3) expressing \( C_i \) as functions of \( Z \). The resulting equation has nine empirical constants.

### References

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