Lecture 12
Models for heavy-ion collisions:
thermal models,
hydrodynamical models,
transport models
Basic models for heavy-ion collisions

• Statistical models:
  basic assumption: the system is described by a (grand) canonical ensemble of non-interacting fermions and bosons in thermal and chemical equilibrium
  [ -: no dynamics]

• (Ideal-) hydrodynamical models:
  basic assumption: conservation laws + equation of state; assumption of local thermal and chemical equilibrium
  [ -: simplified dynamics]

• Transport models:
  based on transport theory of relativistic quantum many-body systems - off-shell Kadanoff-Baym equations for the Green-functions $S^<_{h}(x,p)$ in phase-space representation. Actual solutions: Monte Carlo simulations with a large number of test-particles
  [+: full dynamics | -: very complicated]

Microscopic transport models provide a unique dynamical description of nonequilibrium effects in heavy-ion collisions
Models of heavy-ion collisions

- Initial
- Transport
- Hydro
- Thermal + Expansion
- Thermal model
- Final
Thermal models: ensembles

In equilibrium statistical mechanics one normally encounters three type of ensembles:

- **microcanonical ensemble**: used to describe an isolated system with fixed $E$, $N$, $V$

- **canonical ensemble**: used to describe a system in contact with a heat bath (presumed to represent an infinite reservoir) with fixed $T$, $N$, $V$

- **grand canonical ensemble**: used to describe a system in contact with a heat bath, with which it can exchange particles: with fixed $T$, $V$, $\mu$

In a relativistic quantum system, where particles can be created and destroyed, one computes observables conventionally in the grand canonical ensemble.

Here: $E$ – energy of the system, $N$ – particle number in the system, $V$ – volume of the system, $T$ – temperature of the system, $\mu$ – chemical potential.
The grand canonical partition function reads:

\[ Z = Tr \left[ e^{-\beta (\hat{H} - \mu_i \hat{N}_i)} \right] \]  \hspace{1cm} (1)

with \( \hat{H} \) - Hamiltonian of the system, \( \hat{N}_i \) set of conserved number operators and \( \beta = \frac{1}{T} \).

All other standard thermodynamic properties can be determined from the partition function \( Z = Z(V,T,\mu_1,\mu_2,...) \):

- **Pressure:** \( P = T \frac{\partial \log Z}{\partial V} \)

- **Entropy:** \( S = \frac{\partial T \log Z}{\partial T} \)

- **Particle number:** \( \hat{N}_i = T \frac{\partial \log Z}{\partial \mu_i} \) \hspace{1cm} (2)

The 1st law of thermodynamics

\[ E = -PV + TS + \mu_i N_i \]
Consider a non-interacting gas of bosons or fermions of the same type. The many-body eigenstate $\Psi$ is specified by the single-particle levels $\epsilon_i$ and the set of occupation numbers $n_i$ where $i$ runs over all basis states:

$$N_\Psi = \sum_i n_i, \quad E_\Psi = \sum_i \epsilon_i n_i$$

Summing over all states $\Psi$ gives

$$Z(T,V,\mu) = \sum_{n_1,n_2,...} e^{-\beta \sum_i (\epsilon_i - \mu) n_i} = \sum_{n_1,n_2,...} \prod_i e^{-\beta (\epsilon_i - \mu) n_i} = \prod_i \left\{ \sum_n e^{-\beta (\epsilon_i - \mu) n_i} \right\}$$

For the fermions: the occupation numbers $n$ can only be 0 or 1 $\Rightarrow$ the partition function is

$$Z_F = \prod_i (1 + e^{-\beta (\epsilon_i - \mu) n_i})$$

For the bosons: any occupation number $n$ is allowed $\Rightarrow$ the partition function is

$$Z_B = \prod_i \left( \sum_{n=0}^{\infty} e^{-\beta (\epsilon_i - \mu) n_i} \right) = \prod_i \frac{1}{1 - e^{-\beta (\epsilon_i - \mu)}}$$
Replacing the sum over the eigenstates by an integral \( \sum_i \rightarrow V(2\pi\hbar)^{-3} \int d^3p \) the partition function for either fermions (\( + \)) or bosons (\( - \)) becomes

\[
\ln Z = gV \int \frac{d^3p}{(2\pi\hbar)^3} \ln(1 \pm e^{-\beta(\varepsilon(p) - \mu)})^{\pm 1} \tag{7}
\]

We have added in (7) \( g \) – a degeneracy factor to account for degenerate single-particle levels.

- **Pressure:**
  \[
P = \frac{\partial T \ln Z}{\partial V} = gT \int \frac{d^3p}{(2\pi\hbar)^3} \ln(1 \pm e^{-\beta(\varepsilon(p) - \mu)})^{\pm 1} \tag{8}
\]

- **Particle number:**
  \[
  N = \frac{\partial T \ln Z}{\partial \mu} = gV \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{-\beta(\varepsilon(p) - \mu)}}^{\pm 1} \tag{9}
  \]

- **Energy:**
  \[
  E = -PV + TS + \mu N = gV \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\varepsilon(p)}{e^{-\beta(\varepsilon(p) - \mu)}}^{\pm 1} \tag{10}
  \]

The dispersion relation for single-particle energy:
\[
\varepsilon(p) = \sqrt{p^2 + m^2} \tag{11}
\]
Thermal model: thermal fit

- Particle density for the hadrons of type \( i \)

\[
\rho_i = \frac{N_i}{V} = g \int \frac{d^3 p}{(2\pi \hbar)^3} \frac{1}{e^{-\beta(\varepsilon_i(p)-\mu_i)} \pm 1}
\]  

\( i \) – particle species

Thermal fit of the experimental data –

from particle abundances \( (N_i) \) or particle ratios (in order to exclude the volume \( V \)) one can find the Lagrange parameters \( T, \mu \)

Good description of the hadron abundances by the thermal hadron gas model ➔

The hadron abundances are in rough agreement with a thermally equilibrated system !

➔ Partial equilibration is approximately reached in central heavy-ion collisions at high energies!
Thermal model: thermal fits

The phase diagram of QCD

From the thermal fits of the particle ratios

⇒ chemical freeze-out line $T(\mu)$

⇒ phase boundary – phase transition from hadronic matter to a quark-gluon-plasma
Ideal hydrodynamics

Basic ideas:
- Assumption of local thermal and chemical equilibrium
- Conservation laws + equation of state

- For an isotropic system in local equilibrium, in the local rest frame (LR) of the fluid, the energy momentum tensor and conserved currents

\[
T_{LR}^{\mu\nu} = \begin{pmatrix}
\varepsilon & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{pmatrix}, \quad N_i^{\mu},LR = (n_i, \vec{0}) \quad [i = 1, ..., I]
\]  \hspace{1cm} (13)

are given by a few, local, macroscopic variables: i.e. energy density \( \varepsilon(x) \), pressure \( p(x) \), charge densities \( n_i(x) \)

- In an arbitrary frame where the fluid could be moving

\[
T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p g^{\mu\nu}, \quad N_i^{\mu} = n_i u^\mu
\]  \hspace{1cm} (14)

where \( u^\mu(x) = \frac{1}{\sqrt{1-v^2(x)}}(1, \vec{u}(x)) \) is the four-velocity of the fluid \( (u^\mu u_\mu = 1) \)
Ideal hydrodynamics

\[ u_\mu(x) \] is fixed by energy-momentum and charge conservation

\[ \partial_\mu T^{\mu\nu}(x) = 0, \quad \partial_\mu N^\mu_i(x) = 0 \quad [i = 1, \ldots, I] \]  

(15)

4 + I equations for 5 + I unknowns \((\varepsilon, p, u^\mu, \{n_i\})\)

(in heavy-ion physics, typically \(I \leq 1\), we only follow baryon charge \(n_B\), if at all)

\[
\begin{bmatrix}
T_{00} & T_{01} & T_{02} & T_{03} \\
T_{10} & T_{11} & T_{12} & T_{13} \\
T_{20} & T_{21} & T_{22} & T_{23} \\
T_{30} & T_{31} & T_{32} & T_{33}
\end{bmatrix}
\]

(16)

\(p(\varepsilon, \{n_i\})\)

- Missing ingredient: equation of state
- The set of partial differential equations (15) can then be solved for given initial conditions and freeze-out conditions
Ideal hydrodynamics: initial conditions

Initial conditions:
(for each individual cell of the fluid)

Typical ingredients:
- thermalization time $\tau_0$
- shape of initial entropy (or energy) density profile
- maximum entropy (or energy) density $s_0$ (or $\varepsilon_0$)
- baryon density to entropy ratio $n_B/s$ (assumed to be constant)
- initial radial flow is typically set to zero

Parameters $\tau_0$, $s_0$ (or $\varepsilon_0$) $n_B/s$ and also freezeout parameters $T_{f0}$ ($\varepsilon_{f0}$), are fitted to data for central collisions. Noncentral collisions can then be predicted.
Ideal hydrodynamics: EoS

Lattice QCD EoS - equation of state

Bag-model EoS - equation of state

In the hadronic phase - a resonance gas equation of state is taken, while in the plasma phase $p = (\varepsilon - 4B)/3$, where $B = (0.23 \text{ GeV})^4$ is the bag constant.
Cooper-Frye freezeout:

sudden transition from the fluid to a gas on a 3D hypersurface

(typically $T_{f_0} = \text{const}$ or $\varepsilon_{f_0} = \text{const}$), where $f$ is the phase-space density of a local equilibrium gas boosted with velocity $u(x)$:

$$E dN = p^\mu d\sigma_\mu(x) d^3 p f_{\text{gas}}(x, \vec{p})$$

where $d\sigma_\mu$ is the hypersurface normal to $x$

Application:

- evolve hydro until the end of the phase transition (for the EoS with a QGP)
- then do Cooper-Frye - assume a noninteracting hadron gas
- decay unstable resonances ($T_{f_0}$ ($\varepsilon_{f_0}$) and fit to data)
Ideal hydrodynamics

Success of ideal hydro - spectra

Kolb & Heinz, nucl-th/0305084

Au+Au at 200 GeV
2+1D ideal hydro
bag EOS
75% wounded
+ 25% binary

\[ \tau_0 = 0.6 \text{ fm} \]
\[ s_0 = 110 / \text{ fm}^3 \]
\[ \varepsilon_0 = 25 \text{ GeV/fm}^3 \]
\[ T_0 = 360 \text{ MeV} \]
\[ \varepsilon_{fo} = 0.075 \text{ GeV/fm}^3 \]
\[ T_{fo} \approx 130 \text{ MeV} \]
Ideal hydrodynamics: collective flow

Success of ideal hydro – $v_2$

- $v_2$ versus centrality
  Ideal hydro (open boxes) reproduces the integrated charge particle $v_2$ data (black dots) for impact parameters $b < 7$ fm. In peripheral collisions it overshoots $v_2$ - the smaller and more dilute the system is, the more difficult for it to thermalize.

- $v_2$ versus $p_T$
  In the bulk (low $p_T$): hydrodynamics works! (full hydro or blastwave parametrization) ➔
  - The system behaves like a strongly interacting liquid (of low viscosity).
  - The system is likely to be partonic but not ‘plasma-like’ (weakly interacting).
Dynamical transport models

- Dynamics of heavy-ion collisions – a many-body problem!

- General Schrödinger equation:

\[
i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) = \hat{H}\Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t)
\]

many-body wave-function \quad many-body Hamiltonian

- Approximation - time dependent Hartree-Fock (TDHF) theory:

- many-body wave-function \Rightarrow antisymmetrized product of single particle wave functions

\[
\Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N, t) \Rightarrow A \prod_{\alpha=1}^{N} \psi_{\alpha}(\vec{r}_\alpha, t)
\]

- many-body Hamiltonian \Rightarrow kinetic energy + potential with two-body interactions:

\[
\hat{H}(\vec{r}_1, \vec{r}_2, ...\vec{r}_N, t) \Rightarrow \sum_{\alpha} \hat{T}_{\alpha}(\vec{r}_\alpha, t) + \frac{1}{2} \sum_{\alpha \neq \beta}^{N} \hat{V}_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta, t)
\]

\text{non-local} \quad \text{local}
**Dynamical transport model**

TDHF equation for single-particle state $\alpha$:

$$i\hbar \frac{\partial}{\partial t} \psi_\alpha(\vec{r},t) = h(\vec{r},t)\psi_\alpha(\vec{r},t)$$  \hspace{1cm} (21)

where $h(r,t)$ is the single-particle Hartree-Fock Hamiltonian:

$$h(\vec{r},t) = T(\vec{r}) + \sum_{\beta_{occ}} \int d^3 r' \psi_\beta^*(\vec{r}',t) V(\vec{r} - \vec{r}',t) \psi_\beta(\vec{r}',t)$$  \hspace{1cm} (22)

*in (22) we neglected the exchange (Fock) term (to reduce complexity in notation)*

Kinetic energy:  $T(\vec{r}) = -\frac{\hbar^2}{2m} \nabla_r^2$

Introduce the single particle density matrix:

$$\rho(\vec{r},\vec{r}',t) = \sum_{\beta_{occ}} \psi_\beta^*(\vec{r}',t) \psi_\beta(\vec{r},t)$$  \hspace{1cm} (23)

Thus, (22) can be written as

$$h(\vec{r},t) = T(\vec{r}) + \sum_{\beta_{occ}} \int d^3 r' V(\vec{r} - \vec{r}',t) \rho(\vec{r}',\vec{r}',t)$$  \hspace{1cm} (24)
Dynamical transport model

Multiply (1) by:

\[ \psi_\alpha^*(\vec{r}',t) \cdot (21) : \]

\[ \psi_\alpha^*(\vec{r}',t) i\hbar \frac{\partial}{\partial t} \psi_\alpha(\vec{r},t) = \psi_\alpha^*(\vec{r}',t) h(\vec{r},t) \psi_\alpha(\vec{r},t) \]  

(25)

\[ (21)^+ |_{for \vec{r}' \cdot \psi_\alpha(\vec{r}',t)} : \]

\[ - i\hbar \left[ \frac{\partial}{\partial t} \psi_\alpha^*(\vec{r}',t) \right] \psi_\alpha(\vec{r},t) = h(\vec{r},t) \psi_\alpha^*(\vec{r}',t) \psi_\alpha(\vec{r},t) \]  

(26)

\[ \sum_\alpha ((25)-(26)) : \]

\[ i\hbar \frac{\partial}{\partial t} \rho(\vec{r},\vec{r}',t) = [h(\vec{r},t) - h(\vec{r}',t)] \rho(\vec{r},\vec{r}',t) \]  

(27)

\[ \rho(\vec{r},\vec{r}',t) = \sum_{\beta_{\text{occ}}} \psi_\beta^*(\vec{r}',t) \psi_\beta(\vec{r},t) \]

\[ h(\vec{r},t) = T(\vec{r}) + \sum_{\beta_{\text{occ}}} \int d^3 r' V(\vec{r} - \vec{r}',t) \rho(\vec{r}',\vec{r}',t) \]

\[ = T(\vec{r}) + U(\vec{r},t) \]

kinetic term + potential (local) term
Rewrite (27) using $x$ instead of $r$

\[
\frac{\partial}{\partial t} \rho(\bar{x}, \bar{x}', t) + \frac{i}{\hbar} \left[ \frac{\hbar}{2m} \bar{\nabla}_x^2 + U(\bar{x}, t) - \frac{\hbar}{2m} \bar{\nabla}_{x'}^2 - U(\bar{x}', t) \right] \rho(\bar{x}, \bar{x}', t) = 0
\]  

(28)

Instead of considering the density $\rho(\bar{x}, \bar{x}', t)$, let’s find the equation of motion of its Fourier transform, i.e. Wigner transform density:

\[
f(\bar{r}, \bar{p}, t) = \int d^3s \exp \left( -\frac{i}{\hbar} \bar{p} \bar{s} \right) \rho \left( \bar{r} + \frac{\bar{s}}{2}, \bar{r} - \frac{\bar{s}}{2}, t \right)
\]  

(29)

New variables:

old : $\bar{x}$ $\bar{x}'$

\[
\bar{r} = \frac{\bar{x} + \bar{x}'}{2}, \quad \bar{s} = \bar{x} - \bar{x}'
\]

$f(\bar{r}, \bar{p}, t)$ is the single particle phase-space distribution function

Density in coordinate space:

\[
\rho(\bar{r}, t) = \frac{1}{(2\pi\hbar)^3} \int d^3p \ f(\bar{r}, \bar{p}, t)
\]  

(30)

Density in momentum space:

\[
g(\bar{p}, t) = \int d^3r \ f(\bar{r}, \bar{p}, t)
\]  

(31)
Dynamical transport model

Make Wigner transformation of eq.(28)

\[
\int d^3 s \exp\left(-\frac{i}{\hbar} \vec{p} \vec{s}\right) \frac{\partial}{\partial t} \rho\left(\vec{r} + \frac{\vec{s}}{2}, \vec{r} - \frac{\vec{s}}{2}, t\right)
\]

\[
+ \frac{i}{2m} \hbar^2 \int d^3 s \exp\left(-\frac{i}{\hbar} \vec{p} \vec{s}\right) \left[ \nabla^2 \left(\vec{r} + \frac{\vec{s}}{2}\right) - \nabla^2 \left(\vec{r} - \frac{\vec{s}}{2}\right) \right] \rho\left(\vec{r} + \frac{\vec{s}}{2}, \vec{r} - \frac{\vec{s}}{2}, t\right)
\]

\[
+ \frac{i}{\hbar} \int d^3 s \exp\left(-\frac{i}{\hbar} \vec{p} \vec{s}\right) \left[ U\left(\vec{r} + \frac{\vec{s}}{2}, t\right) - U\left(\vec{r} - \frac{\vec{s}}{2}, t\right) \right] \rho\left(\vec{r} + \frac{\vec{s}}{2}, \vec{r} - \frac{\vec{s}}{2}, t\right) = 0
\]

- Use that \(\nabla^2 \frac{\vec{s}}{2} - \nabla^2 \frac{\vec{s}}{2} = 2\nabla \cdot \nabla \vec{s}\) (33)

- Consider \(U\left(\vec{r} + \frac{\vec{s}}{2}, t\right) - U\left(\vec{r} - \frac{\vec{s}}{2}, t\right)\)

Make Taylor expansion around \(r; s \to 0\)

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{2} \vec{s} \cdot \nabla \vec{r}\right)^n U \mid_{s=0} - \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2} \vec{s} \cdot \nabla \vec{r}\right)^n U \mid_{s=0} = 2 \sum_{odd}^{\infty} \frac{1}{2} \vec{s} \cdot \nabla \vec{r} \right)^n U \mid_{s=0}
\]

\[\approx \vec{s} \cdot \nabla \vec{r} U(\vec{r}, t)\]

Classical limit: keep only the first term \(n=1\)
From (32) and (33), (34) obtain

\[
\begin{align*}
\frac{\partial}{\partial t} f(\mathbf{r}, \mathbf{p}, t) + \frac{i}{2m} \frac{\hbar^2}{\hbar} \int d^3s \exp\left(-\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}\right) 2\mathbf{\nabla}_r \cdot \mathbf{\nabla}_{\mathbf{s}} \rho\left(\mathbf{r} + \frac{s}{2}, \mathbf{r} - \frac{s}{2}, t\right) \\
+ \frac{i}{\hbar} \int d^3s \exp\left(-\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{s}\right) \mathbf{s} \cdot \mathbf{\nabla}_r U(\mathbf{r}, t) \rho\left(\mathbf{r} + \frac{s}{2}, \mathbf{r} - \frac{s}{2}, t\right) &= 0
\end{align*}
\]

(35)

Vlasov equation
- free propagation of particles in the self-generated HF mean-field potential:

\[
\begin{align*}
\frac{\partial}{\partial t} f(\mathbf{r}, \mathbf{p}, t) + \frac{\mathbf{p}}{m} \mathbf{\nabla}_r f(\mathbf{r}, \mathbf{p}, t) - \mathbf{\nabla}_r U(\mathbf{r}, t) \mathbf{\nabla}_p f(\mathbf{r}, \mathbf{p}, t) &= 0
\end{align*}
\]

(36)

Eq. (36) is entirely classical (lowest order in \(s\) expansion).
Here \(U\) is a self-consistent potential associated with \(f\) phase-space distribution:

\[
U(\mathbf{r}, t) = \frac{1}{(2\pi \hbar)^3} \sum_{\beta_{occ}} \int d^3r' d^3p V(\mathbf{r} - \mathbf{r}', t) f(\mathbf{r}', \mathbf{p}, t)
\]

(37)
Dynamical transport model

Eq.(36) is equivalent to:

\[ \frac{d}{dt} f(\vec{r}, \vec{p}, t) = 0 = \left[ \frac{\partial}{\partial t} + \vec{r} \vec{\nabla}_{\vec{r}} + \vec{p} \vec{\nabla}_{\vec{p}} \right] f(\vec{r}, \vec{p}, t) = 0 \]  

\[ (38) \]

Classical equations of motion:

\[ \begin{align*}
\dot{\vec{r}} &= \frac{d\vec{r}}{dt} = \frac{\vec{p}}{m} \\
\dot{\vec{p}} &= \frac{d\vec{p}}{dt} = -\vec{\nabla}_{\vec{r}} U(\vec{r}, t)
\end{align*} \]

\[ (39) \]

Note: the quantum physics only plays a role in the initial conditions for \( f \): the initial \( f \) must respect the Pauli principle, however, the identity of particles plays no role beyond the initial conditions.

Trajectory: \( \vec{r}(t) \)

1

2