Lecture 5

Coupling of Angular Momenta
Isospin
Nucleon-Nucleon Interaction

I. Angular Momentum Operator

Rotation $R(\theta)$: in polar coordinates the point $r = (r, \phi)$ is transformed to $r' = (r, \phi + \theta)$ by a rotation through the angle $\theta$ around the z-axis.

Define the rotated wave function as: $\psi'(r') = \psi(r)$

Its value at $r$ is determined by the value of $\phi$ at that point, which is transformed to $r$ by the rotation:

$R(\theta) \psi(r, \phi) = \psi'(r, \phi) = \psi(r, \phi - \theta)$

The shift in $\phi$ can also be expressed by a Taylor expansion

$$\psi'(r, \phi) = \sum_{n=0}^{\infty} \frac{(-\theta)^n}{n!} \frac{\partial^n}{\partial \phi^n} \psi(r, \phi)$$

Rotation (in exponential representation):

$$R(\theta) = \exp \left( -\frac{i}{\hbar} \theta \hat{J}_z \right) \approx \left( 1 - \frac{i}{\hbar} \theta \hat{J}_z \right)$$

where the operator $J_z$ for infinitesimal rotations is the angular-momentum operator:

$$\hat{J}_z = -i\hbar \frac{\partial}{\partial \phi}$$
Angular Momenta

- Cartesian coordinates \((x,y,z)\)
- Finite rotations about any of these axes may be written in the exponential representation as
  \[ \mathcal{R}(\theta_k) = \exp \left( -\frac{i}{\hbar} \theta_k \hat{J}_k \right) , \quad k \in \{x, y, z\} \]

- The cartesian form of the angular-momentum operator \(\hat{J}_k\) reads
  \[\hat{J}_k = -i\hbar \left. \frac{\partial \mathcal{R}(\theta_k)}{\partial \theta_k} \right|_{\theta_k=0} \rightarrow \begin{align*}
  \hat{J}_x &= -i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) , \\
  \hat{J}_y &= -i\hbar \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) , \\
  \hat{J}_z &= -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) . \end{align*}\]

- With commutation relations (SU(2) algebra):
  \[ [J_x, J_y] = i\hbar J_z , \quad [J_y, J_z] = i\hbar J_x , \quad [J_z, J_x] = i\hbar J_y \quad [J_k, J^2] = 0 \]

- Consider representation using the basis \(|jm\rangle\) diagonal in both \(\hat{J}^2\) and \(\hat{J}_z\)
  \[\hat{J}^2 \mid jm\rangle = \hbar^2 \Lambda_j \mid jm\rangle , \quad \hat{J}_z \mid jm\rangle = \hbar m \mid jm\rangle \]
  with eigenvalue \(\Lambda_j = j(j+1)\).
Coupling of Angular Momenta

A system of two particles with angular momenta $\hat{J}_1$ and $\hat{J}_2$ has the total angular momentum $\hat{J} = \hat{J}_1 + \hat{J}_2$

Eigenfunctions of $\hat{J}_1$ and $\hat{J}_2$ (and $\hat{J}_{1z}$ and $\hat{J}_{2z}$) are: $|j_1,m_1\rangle$ and $|j_2,m_2\rangle$

\begin{align*}
\hat{J}_1^2 |j_1 m_1\rangle &= \hbar^2 j_1 (j_1 + 1) |j_1 m_1\rangle, \\
\hat{J}_{1z} |j_1 m_1\rangle &= \hbar m_1 |j_1 m_1\rangle, \\
\hat{J}_2^2 |j_2 m_2\rangle &= \hbar^2 j_2 (j_2 + 1) |j_2 m_2\rangle, \\
\hat{J}_{2z} |j_2 m_2\rangle &= \hbar m_2 |j_2 m_2\rangle.
\end{align*} 

(1)

A basis for the system of two particles may then be built out of the products of these states, forming the so-called uncoupled basis states

$$ |j_1 m_1 j_2 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle $$ 

(2)

Such a state is an eigenstate of the $z$ component of the total angular momentum $J_z$ with eigenvalue $m_1 + m_2$, since

$$ \hat{J}_z |j_1 m_1 j_2 m_2\rangle = (\hat{J}_{1z} + \hat{J}_{2z}) |j_1 m_1 j_2 m_2\rangle = \hbar (m_1 + m_2) |j_1 m_1 j_2 m_2\rangle $$ 

(3)

however, it can not be an eigenstate of $\hat{J}^2$, since $\hat{J}^2$ doesn't commute with $\hat{J}_{1z}$ and $\hat{J}_{2z}$

$$ [\hat{J}^2, \hat{J}_{1z}] = [\hat{J}_1^2, \hat{J}_{1z}] + [\hat{J}_2^2, \hat{J}_{1z}] + [2\hat{J}_1 \cdot \hat{J}_2, \hat{J}_{1z}] = 2i\hbar(\hat{J}_{2y}\hat{J}_{1x} - \hat{J}_{2x}\hat{J}_{1y}) $$

$$ [\hat{J}_z, \hat{J}_1^2] = [\hat{J}_z, \hat{J}_2^2] = [\hat{J}^2, \hat{J}_1^2] = [\hat{J}^2, \hat{J}_2^2] = 0 $$

(4)
Coupling of Angular Momenta

- A fully commuting set of operators is thus only given by $\hat{J}_z, \hat{J}_1^2, \hat{J}_2^2$ and $\hat{J}_2^2$.

- Angular-momentum coupling: the new basis vectors $| jm_j, j_2>$ can be expanded in the uncoupled basis states $|j_1 m_1, j_2 m_2>$ as

$$
| jm_j, j_2> = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1 m_1, j_2 m_2 | jm \rangle | j_1 m_1, j_2 m_2 >
$$

(4)

The expansion coefficients are denoted as Clebsch-Gordon-coefficients:

$$
\langle j_1 m_1, j_2 m_2 | jm_j, j_2 > = \langle jm_j, j_2 | j_1 m_1, j_2 m_2 > = \langle j_1 j_2 j | m_1 m_2 m \rangle = \langle j_1 m_1, j_2 m_2 | jm \rangle = C_{j_1 m_1, j_2 m_2}^{jm}
$$

The name derives from the German mathematicians Alfred Clebsch (1833–1872) and Paul Gordan (1837–1912).

- Properties of Clebsch-Gordon-coefficients:

  - Selection rules: the coefficient is zero unless the quantum numbers fulfil the two conditions

$$
m_1 + m_2 = m \quad m = -j, \ldots, j
$$

$$
| j_1 - j_2 | \leq j \leq | j_1 + j_2 |
$$

(5)

"triangular condition": the size of the total angular momentum is restricted to those values that are allowed by the rules of vector addition, where the vectors $\hat{J}_1, \hat{J}_2$ and $\hat{J}$ form a triangle.
Properties of Clebsch-Gordon coefficients

- Clebsch-Gordon coefficients are real \( \langle j_1 m_1, j_2 m_2 | jm \rangle = \langle jm | j_1 m_1, j_2 m_2 \rangle \)

- Transformation back to the uncoupled basis:

\[
| j_1 m_1, j_2 m_2 \rangle = \sum_{|j_1+j_2|} \sum_{j=|j_1-j_2|}^{j} \sum_{m=-j} \langle j_1 m_1, j_2 m_2 | jm \rangle | jm j_1, j_2 \rangle
\]

(6)

- Orthogonality relations: from (4) and (6) and the completeness relation

\[
1 \equiv \sum_{x} \langle x | x \rangle \\
\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1 m_1, j_2 m_2 | jm \rangle \langle j_1 m_1, j_2 m_2 | j' m' \rangle = \delta_{j'j} \delta_{mm'}
\]

\[
| j_1 + j_2 \rangle \sum_{j=|j_1-j_2|}^{j} \sum_{m=-j} \langle j_1 m_1, j_2 m_2 | jm \rangle \langle j_1 m_1', j_2 m_2' | jm \rangle = \delta_{m,m'} \delta_{m,m'}
\]

- Special cases: for \( j=0 \) \( \langle j_1 m_1, j_2 m_2 | 00 \rangle = \delta_{j_1,j_2} \delta_{m_1,-m_2} \frac{(-1)^{j_1-m_1}}{\sqrt{2j_2 + 1}} \)

for \( j=j_1+j_2 \) and \( m=j \)

\[
\langle j_1 j_1, j_2 j_2 | (j_1 + j_2)(j_1 + j_2) \rangle = 1
\]
Properties of Clebsch-Gordon coefficients

- Symmetry properties:

\[
\begin{align*}
\langle j_1 m_1 j_2 m_2 | J M \rangle &= (-1)^{j_1+j_2-J} \langle j_1 -m_1 j_2 -m_2 | J -M \rangle \\
&= (-1)^{j_1+j_2-J} \langle j_2 m_2 j_1 m_1 | J M \rangle \\
&= (-1)^{j_1-m_1} \sqrt{\frac{2J+1}{2j_2+1}} \langle j_1 m_1 J -M | j_2 -m_2 \rangle \\
&= (-1)^{j_2+m_2} \sqrt{\frac{2J+1}{2j_1+1}} \langle J -M j_2 m_2 | j_1 -m_1 \rangle \\
&= (-1)^{j_1-m_1} \sqrt{\frac{2J+1}{2j_2+1}} \langle J M j_1 -m_1 | j_2 m_2 \rangle \\
&= (-1)^{j_2+m_2} \sqrt{\frac{2J+1}{2j_1+1}} \langle j_2 -m_2 J M | j_1 m_1 \rangle
\end{align*}
\]
Wigner 3-j symbols

- **Relation** of Clebsch-Gordon-coefficients to Wigner 3-j (or 3-jm) symbols:

\[ \langle j_1 m_1 j_2 m_2 | j_3 m_3 \rangle = (-1)^{j_1-j_2+m_3} \sqrt{2j_3+1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \]

- **Symmetry properties:**
  - 3j symbol is **invariant** under an **even permutation** of its columns:
    \[ \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} \]
  - **An odd permutation** of the columns gives a **phase factor**:
    \[ \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} \]
  - Changing the sign of the \( m \) quantum numbers also gives a phase factor:
    \[ \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \]

- **Selection rules**
  The Wigner 3-j is zero unless all these conditions are satisfied:
  \[ m_1 + m_2 + m_3 = 0 \]
  \[ j_1 + j_2 + j_3 \text{ is an integer} \]
  \[ |m_i| \leq j_i \]
  \[ |j_1 - j_2| \leq j_3 \leq j_1 + j_2. \]
**Wigner 3-j symbols**

- **Scalar invariant**
  The contraction of the product of three rotational states with a $3j$ symbol:
  \[
  \sum_{j_1} \sum_{j_2} \sum_{j_3} |j_1 m_1 \rangle |j_2 m_2 \rangle |j_3 m_3 \rangle \binom{j_1}{j_2 \ j_3} \binom{j_1 \ j_2 \ j_3}{m_1 \ m_2 \ m_3}
  \]
  is invariant under rotations.

- **Orthogonality relations:**
  \[
  (2j + 1) \sum_{m_1 m_2} \binom{j_1}{j_2 \ j} \binom{j_1 \ j_2 \ j'}{m_1 \ m_2 \ m'} = \delta_{jj'} \delta_{m m'}.
  \]
  \[
  \sum_{j m} (2j + 1) \binom{j_1}{j_2 \ j m} \binom{j_1 \ j_2 \ j}{j_{m_1} \ m_2 \ m} = \delta_{m_1 m'} \delta_{m_2 m'}.
  \]

- **Relation to spherical harmonics:**
  The $3jm$ symbols give the integral of the products of three spherical harmonics:
  \[
  \int Y_{l_1 m_1} (\theta, \varphi) Y_{l_2 m_2} (\theta, \varphi) Y_{l_3 m_3} (\theta, \varphi) \sin \theta \, d\theta \, d\varphi
  = \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}} \binom{l_1 \ l_2 \ l_3}{l_1 \ l_2 \ l_3} \binom{l_1 \ l_2 \ l_3}{m_1 \ m_2 \ m_3}
  \]
Wigner D-matrices

Relation of Clebsch-Gordon-coefficients to Wigner D-matrices

\[
\int_0^{2\pi} d\alpha \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\gamma D^j_{MK} (\alpha, \beta, \gamma)^* D^{j_1}_{m_1 k_1} (\alpha, \beta, \gamma) D^{j_2}_{m_2 k_2} (\alpha, \beta, \gamma) = \frac{8\pi^2}{2J+1} \langle j_1 m_1 j_2 m_2 | J M \rangle \langle j_1 k_1 j_2 k_2 | J K \rangle
\]

The Wigner D-matrix is a square matrix of dimension \(2j + 1\) with matrix elements defined by the rotation operator within spherical harmonics:

\[
D^j_{m'm} (\alpha, \beta, \gamma) \equiv \langle jm' | \mathcal{R}(\alpha, \beta, \gamma) | jm \rangle = e^{-im'\alpha} d^j_{m'm} (\beta) e^{-im\gamma}
\]

where the matrix (with elements)

\[
d^j_{m'm} (\beta) = \langle jm' | e^{-i\beta j_y} | jm \rangle
\]

is known as Wigner's (small) d-matrix

The rotation operator reads as

\[
\mathcal{R}(\alpha, \beta, \gamma) = e^{-i\alpha j_z} e^{-i\beta j_y} e^{-i\gamma j_z}
\]

where \(\alpha, \beta, \gamma\) are the Euler angles

Explicit result for Wigner's (small) d-matrix:

\[
d^j_{m'm} (\beta) = \frac{(-1)^{m'-m+s}}{(j+m-s)! (j'-m+s)! (j-m-s)!} \sum_s \frac{(2j+1)! (j'-m+s)! (j-m-s)!}{(j+m-s)! (j'-m-s)! (j-m-s)!} \times \left( \cos \frac{\beta}{2} \right)^{2j+m-m'-2s} \left( \sin \frac{\beta}{2} \right)^{m'-m+2s}
\]

The sum over \(s\) is over such values that the factorials are non-negative.
**Table of Clebsch-Gordan coefficients, Spherical Harmonics, and d Functions**

\[
\begin{aligned}
\text{Notation: } & & \begin{array}{cccc}
J & J & \ldots \\
M & M & \ldots \\
\ldots & \ldots & \ldots
\end{array} \\
\text{Coefficients: } & & \begin{array}{cccc}
m_1 & m_2 \\
m_1 & m_2 \\
\ldots & \ldots
\end{array}
\end{aligned}
\]

\[
Y_j^m = (-1)^m Y_{jm+}
\]

\[
\begin{align*}
Y_1^0 &= \frac{1}{\sqrt{4\pi}} \cos \theta \\
Y_1^1 &= \frac{-1}{\sqrt{8\pi}} \sin \phi e^{i\phi} \\
Y_2^0 &= \frac{1}{2\sqrt{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \\
Y_2^1 &= \frac{-1}{\sqrt{8\pi}} \sin \cos \phi e^{i\phi} \\
Y_2^2 &= \frac{1}{2\sqrt{4\pi}} \sin \theta e^{2i\phi}
\end{align*}
\]

\[
\langle j_1j_2m_1m_2|j_1j_2JM \rangle = (-1)^J \delta_{j_1-j_2} \langle j_2j_1m_2m_1|j_2j_1JM \rangle
\]

\[
\begin{array}{cccc}
1/2 \times 1/2 & 1/2 \times 1/2 & 2 \times 1/2 & 3/2 \times 1/2 \\
1/2 \times 1/2 & 1/2 \times 1/2 & 2 \times 1/2 & 3/2 \times 1/2
\end{array}
\]
II. Isospin

- The isospin is an internal quantum number used in the classification of elementary particles (analogous to the spin).

Isospin was introduced by Werner Heisenberg in 1932 to explain the fact that the strength of the strong interaction is almost the same between two protons or two neutrons as between a proton and a neutron, unlike the electromagnetic interaction which depends on the electric charge of the interacting particles.

Heisenberg's idea was that protons and neutrons were essentially two states of the same particle, the nucleon, analogous to the 'up' and 'down' states of a spin-1/2 particle, i.e.

- the proton is analogous to the 'spin-up' state and
- the neutron to the 'spin-down' state.

Even if one ignores charge, the proton and neutron are still not completely symmetric since the neutron is slightly more massive.

- isospin is not a perfect symmetry but a good approximate symmetry of the strong interaction.
Isospin

Formally: The isospin symmetry is given by the invariance of the Hamiltonian of the strong interactions under the action of the Lie group SU(2). The neutron and the proton are assigned to the doublet (of the fundamental representation) of SU(2):

\[ |p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

A transformation in SU(2) (2x2 matrix with determinant 1) will then mix proton and neutron states.

- The generators of the SU(2) group for isospin states are the isospin operators

\[ \hat{\tau} = \frac{1}{2} \tau \]

where \( \tau \) is a 3-vector of matrices that is identical to the Pauli vector \( \sigma \) and just has a different notation to make clear that it refers to a different physical degree of freedom (isospin instead of spin):

\[ \tau = \begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} \quad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
Isospin

If a system consists of $A$ nucleons, its total isospin is defined by

$$\hat{T} = \sum_{i=1}^{A} \hat{t}_i$$

A Hamiltonian is isospin invariant if it commutes with the operators $t \rightarrow T$ and $T_z$ are good quantum numbers:

$$\hat{T}^2 |T, T_z\rangle = T(T + 1) |T, T_z\rangle \quad \hat{T}_z |T, T_z\rangle = T_z |T, T_z\rangle$$

As for angular momentum, the eigenvalue of the $z$ projection is simply the sum of the projections for the individual nucleons; since each of these contributes $+1/2$ for protons and $-1/2$ for neutrons, it is given by

$$\hat{T}_z = \sum_{i=1}^{A} \hat{t}_{zi} = \frac{1}{2} (Z - N)$$

Since the strong Hamiltonian is isospin invariant, the isospin shift operator $T^+_z - \hat{T}^+_z |T_z\rangle = |T_z + 1\rangle$ - commute with $H$, i.e.

$$\hat{H} \hat{T}^+_z |T, T_z\rangle = \hat{H} |T, T_z + 1\rangle = E(T, T_z + 1) |T, T_z + 1\rangle$$

$$\hat{T}^+_z \hat{H} |T, T_z\rangle = E(T, T_z + 1) |T, T_z + 1\rangle$$

The eigenstates for different $T_z$ correspond to nuclei with the same number of nucleons, but different proton-to-neutron ratios, but all have the same energy. Existance of analog states in a chain of isobaric nuclei which have the same internal structure. Such states are found in light nuclei where Coulomb effects are small.
Isospin

The proton and neutron must be the eigenstates of \( t_z \) in the two-dimensional representation corresponding to isospin 1/2; whether one chooses the positive eigenvalue for the proton or the neutron is a matter of choice.

By convention: the proton gets the positive isospin projection, so that

\[
\hat{t}_z |p\rangle = \frac{1}{2} |p\rangle \quad \hat{t}_z |n\rangle = -\frac{1}{2} |n\rangle
\]

The operator \( t^2 \) has an eigenvalue \( \frac{3}{4} \):

\[
\hat{t}^2 |p\rangle = \frac{1}{2} \left( \frac{1}{2} + 1 \right) |p\rangle = \frac{3}{4} |p\rangle \quad \hat{t}^2 |n\rangle = \frac{3}{4} |n\rangle
\]

- \( t_x \) and \( t_y \) transform protons into neutrons and vice versa; for example

\[
\hat{t}_x |p\rangle = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} |n\rangle
\]

- Introduce the raising \( t_+ \) and lowering \( t_- \) operators by:

\[
\hat{t}_+ = \hat{t}_x + i\hat{t}_y = \frac{1}{2} \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

\[
\hat{t}_- = \hat{t}_x - i\hat{t}_y
\]

then \( t_+ \) transforms \( |n\rangle \) to \( |p\rangle \) state:

\[
\hat{t}_+ |n\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |p\rangle
\]
Isospin of the two-nucleon system

For the isospin part of the wave functions we have the uncoupled basis consisting of the four states (the indices 1 and 2 refer to the two particles)

\[ |p_1\rangle|p_2\rangle, |p_1\rangle|n_2\rangle, |n_1\rangle|p_2\rangle, |n_1\rangle|n_2\rangle \]

Applying angular momentum results in the coupled basis:

\[
\begin{align*}
T_z &= +1 & |p_1\rangle|p_2\rangle \\
T = 1 & T_z &= 0 & \frac{1}{\sqrt{2}}(|p_1\rangle|n_2\rangle + |n_1\rangle|p_2\rangle) & \text{symmetric} \\
 & T_z &= -1 & |n_1\rangle|n_2\rangle \\
T = 0 & T_z &= 0 & \frac{1}{\sqrt{2}}(|p_1\rangle|n_2\rangle - |n_1\rangle|p_2\rangle) & \text{antisymmetric}
\end{align*}
\]

The states with total isospin \(T=1\) are symmetric under exchange of the two particles, whereas the state with isospin \(T=0\) is antisymmetric.

Isospin invariance of the strong interaction implies that it should not depend on the isospin projection, so that the systems \(p-p\) and \(n-n\) should have similar scattering behavior. For the \(p-n\) system only the symmetric component with isospin 1 should act the same, whereas the antisymmetric component with isospin 0 may have totally different scattering properties. Thus isospin invariance does not predict identical behaviour of the \(p-n\) system to \(p-p\) and \(n-n\)! (modifications due to Coulomb effects are in any case neglected in this discussion).
The nucleon-nucleon potential can depend on the positions, momenta, spins, and isospins of the two nucleons:

$$ v = v(r, r', p, p', \hat{\sigma}, \hat{\sigma}', \hat{r}, \hat{r}') $$

The functional form of $v$ is restricted by the invariance requirements:

- **Translational invariance**: the dependence on the positions $r$ and $r'$ should only be through the relative distance $r-r'$.

- **Gallilei invariance**: the interaction potential should be independent of any transformation to another inertial frame of reference. This demands that the interaction should depend only on the relative momentum $p_{rel} = p - p'$.

- **Rotational invariance**: all terms in the potential should be constructed to have a total angular momentum of zero.

- **Isospin invariance**: the isospin degrees of freedom enter only through the operators $\tau$ and $\tau'$. The only terms that are scalar under rotation in isospin space are those containing no isospin dependence, i.e. the scalar product $\tau \cdot \tau'$, or powers thereof (which can be reduced to the first-order product, so that only this needs to be taken into account).
Nucleon-nucleon interaction

> we may split the total potential into two terms:

\[ u = u(r, p, \sigma, \sigma') + \bar{u}(r, p, \sigma, \sigma') \hat{T} \cdot \hat{T}' \]  

(1)

!!! This separation into two terms according to the isospin dependence is only one of several alternative decompositions, which are all useful in a different context.

> The first alternative is based on total isospin: Two nucleons with isospin 1/2 may couple to a total isospin \( T = 0 \) (singlet) or \( T = 1 \) (triplet). The scalar product of the isospin operators can then be expressed as

\[ \hat{T} \cdot \hat{T}' = 4 \hat{t} \cdot \hat{t}' = 2 \left[ (\hat{t} + \hat{t}')^2 - \hat{t}_1^2 - \hat{t}_2^2 \right] \]

(1.1)

This result allows the construction of projection operators into the singlet or triplet, respectively, which are simply such linear combinations that they yield zero when applied to one of the two states and 1 when applied to the other:

\[ \hat{P}_{T=0} = \frac{1}{4} (1 - \hat{T} \cdot \hat{T}') \quad , \quad \hat{P}_{T=1} = \frac{1}{4} (3 + \hat{T} \cdot \hat{T}') \]
The interaction between the nucleons can now be formulated also as a sum of a singlet and a triplet potential:

$$v = v_{T=0}(r, p, \sigma, \sigma') \hat{P}_{T=0} + v_{T=1}(r, p, \sigma, \sigma') \hat{P}_{T=1}$$

\(\Rightarrow\) **A third formulation** is based on isospin exchange. A two-particle wave function with good isospin is

$$|TT_3\rangle = \sum_{t_3t_3'} \left(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \bar{T} |t_3t_3'T_3\rangle |\frac{1}{2} t_3\rangle |\frac{1}{2} t_3'\rangle\right)$$

Using the symmetry of the Clebsch-Gordan coefficients

$$(j_1j_2J|m_1m_2M) = (-1)^{j_1+j_2-J} (j_2j_1J|m_2m_1M)$$

we see that an exchange of the two isospin projections \(t_3\) and \(t_3'\) corresponds to a change of sign for \(T = 0\) and no change for \(T = 1\). Because of (1.1), the isospin exchange operator \(\hat{P}_{\tau}\) - which produces the correct sign change - can be expressed as

$$\hat{P}_{\tau} = \frac{1}{2} \left(1 + \hat{\tau} \cdot \hat{\tau}'\right)$$

and the isospin dependence of the nucleon-nucleon interaction may be formulated in a third way as:

$$v = v_1(r, p, \sigma, \sigma') + v_2(r, p, \sigma, \sigma') \hat{P}_{\tau}$$
Nucleon-nucleon interaction

- **Parity invariance:** the requirement for the potential is
  \[ v(r, p, \hat{\sigma}, \hat{\sigma}', \tau, \tau') = v(-r, -p, \hat{\sigma}, \hat{\sigma}', \tau, \tau') \]

  which can be fulfilled by using only terms containing an even power of \( r \) and \( p \) together.

- **Time reversal invariance:** it requires
  \[ v(r, p, \hat{\sigma}, \hat{\sigma}', \tau, \tau') = v(r, -p, -\hat{\sigma}, -\hat{\sigma}', \tau, \tau') \]

  so that an even number of \( p \) and \( \sigma \) combined are allowed in each term.

\[
\hat{\sigma} \cdot \hat{\sigma}' , \quad (r \cdot \hat{\sigma})(r \cdot \hat{\sigma}') , \\
\hat{L} \cdot \hat{S} = -i\hbar(r \times \hat{p}) \cdot (\hat{\sigma} + \hat{\sigma}')
\]

Terms (of lowest order) which obey invariance requirements:

All of these can be combined with arbitrary functions of \( r \) and \( p \).
Nucleon-nucleon interaction

General form of the potential:

\[ v = v_0(r) + v_\sigma(r) \hat{\sigma} \cdot \hat{\sigma}' + v_\tau(r) \hat{\tau} \cdot \hat{\tau}' + v_{\sigma \tau}(\hat{\sigma} \cdot \hat{\sigma}') (\hat{\tau} \cdot \hat{\tau}') \]  

or within the formulation using exchange operators (parity, spin, isospin):

\[ v = v_W(r) + v_M \hat{P}_r + v_B \hat{P}_\sigma + v_H \hat{P}_r \hat{P}_\sigma \]  

The indices stand for Wigner, Majorana, Bartlett, and Heisenberg. The last term can be also written with \( \hat{P}_\tau \), but due to the conditions \( \hat{P}_r \hat{P}_\sigma \hat{P}_\tau = -1 \) can be expressed as \( \hat{P}_r \hat{P}_\sigma = -\hat{P}_\tau \)

Another important ingredient of a nucleon-nucleon interaction that was found to be necessary to explain the properties of the deuteron is the tensor force. It contains the term \( (r \cdot \hat{\sigma})(r \cdot \hat{\sigma}') \), but in such a combination that the average over the angles vanishes. The full expression is

\[ S_{12} = (v_0(r) + v_1(r) \hat{\tau} \cdot \hat{\tau}') \left[ \frac{(r \cdot \hat{\sigma})(r \cdot \hat{\sigma}')}{r^2} - \frac{1}{3} \hat{\sigma} \cdot \hat{\sigma}' \right] \]
Nucleon-nucleon scattering data give constrains on the NN-interaction and allow to determine the parameters in the NN-potential.

Some basic features emerging from low-energy nucleon-nucleon scattering are:

- the interaction has a **short range** of about 1 fm,
- within this range, it is **attractive** with a depth of about 40 MeV for the larger distances, whereas
- there is **strong repulsion at shorter distances** < 0.5 fm;
- it depends both on **spin and isospin** of the two nucleons.
The idea of Yukawa that the nucleon-nucleon interaction is mediated by pions - just as the Coulomb interaction is caused by the exchange of (virtual) photons - leads to the one-pion exchange potential (OPEP).

Taking the correct invariance properties of the pion field with respect to spin, isospin, and parity into account and regarding the nucleon as a static source of the pion field leads to a nucleon-nucleon interaction of the form:

\[
u_{\text{OPEP}}(r - r', \hat{\sigma}, \hat{\sigma}', \hat{\tau}, \hat{\tau}') = -\frac{f^2}{4\pi\mu} (\hat{\sigma} \cdot \nabla)(\hat{\sigma}' \cdot \nabla') \frac{e^{-\mu|r-r'|}}{|r-r'|} \quad (7)
\]

The OPEP potential shows some, but not all, features of a realistic nucleon-nucleon interaction:
- it contains spin- and isospin-dependent parts as well as a tensor potential,
- the dominant radial dependence is of Yukawa type.

Other properties, however, show that it is not sufficient:
- there is no spin-orbit coupling and
- there is no short-range repulsion.
Conceptually also it is clear that the OPEP potential cannot be the whole story. The contributions of the exchange of more than one pion and other mesons must modify the interaction.

→ One-boson exchange potentials (OBEP)

The type of interaction term generated by each boson depends on its parity and angular momentum and is summarized in Table:

<table>
<thead>
<tr>
<th>Type of meson</th>
<th>Physical meson</th>
<th>Interaction terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>scalar</td>
<td>“σ meson”</td>
<td>1, ( \hat{L} \cdot \hat{s} )</td>
</tr>
<tr>
<td>pseudoscalar</td>
<td>( \pi, \eta, \eta' )</td>
<td>( S_{12} )</td>
</tr>
<tr>
<td>vector</td>
<td>( \rho, \omega, \phi )</td>
<td>1, ( \hat{\sigma} \cdot \hat{\sigma}' ), ( S_{12} ), ( \hat{L} \cdot \hat{s} )</td>
</tr>
</tbody>
</table>

- for the isovector mesons \( \pi \) and \( \rho \), additional factors \( \hat{\tau} \cdot \hat{\tau}' \) appear in the interaction.
- \( \sigma \)-meson is not seen in experiment: \( \Rightarrow \) it might be a very broad resonance or it represents an approximation to the dominant contribution in two-meson exchange.
- The masses and the coupling constants for all the mesons are fitted to experimental scattering data.
The Hamada-Johnston potential has the general form:

\[ V(r) = V_C(r) + V_T(r) S_{12} + V_{LS}(r) \hat{L} \cdot \hat{s} + V_{LL}(r) L_{12} \]  

\[ L_{12} = \hat{L}^2(\hat{\sigma} \cdot \hat{\sigma}') - \frac{1}{2}[(\hat{\sigma} \cdot \hat{L})(\hat{\sigma}' \cdot \hat{L}) + (\hat{\sigma}' \cdot \hat{L})(\hat{\sigma} \cdot \hat{L})] \]  

The functional form of the radial parts is inspired by the meson exchange potential:

\[ V_C(r) = c_0 \mu c^2 (\hat{r} \cdot \hat{r}') (\hat{\sigma} \cdot \hat{\sigma}') \frac{e^{-\mu r}}{\mu r} \left( 1 + a_c \frac{e^{-\mu r}}{\mu r} + b_c \frac{e^{-2\mu r}}{\mu r^2} \right) \]

Another different approach is used in the Reid soft-core and Reid hard-core potential. They are parametrized differently for the various spin and isospin combinations, for example,

\[ V_{1D}(r) = -\frac{10.6 \text{ MeV}}{\mu r} \left( e^{-\mu r} + 4.939 e^{-2\mu r} + 154.7 e^{-6\mu r} \right) \]

For the hard-core potentials, there is a hard-core radius \( r_c \) (different for each spin and isospin combination), below which the potential is set to infinity.
Effective Interactions

All nucleon-nucleon interactions considered above have been constructed from NN-scattering data, i.e. from free (in vacuum) NN-scattering.

However, in nuclei the interaction is modified by complicated many-body effects to such an extent that it becomes more profitable to employ effective interactions, which by the way do not describe nucleon-nucleon scattering (in vacuum) properly.

Some well-known effective interactions:

- the Gauss potential $v(r) = -V_0 \, e^{-r^2/r_0^2}$,
- the Hulthén potential $v(r) = -V_0 \, \frac{\exp(-r/r_0)}{1 - \exp(-r/r_0)}$,
- the contact potential $v(r) = -V_0 \, \delta(r/r_0)$.

Usually $V_0 \approx 50$ MeV and $r_0 \approx 1, \ldots, 2$ fm are used.

The proper in-medium forces can be derived within Brueckner theory.