Free Electron Model

- The free electron model (Fermi gas) can also be used to calculate a pressure called the degeneracy pressure
- As the Fermi gas is compressed, the energy of the electrons increases and positive pressure must be applied to compress it
  - This degenerate motion is forced on the electrons by quantum mechanics and it cannot be stopped by cooling the solid even to absolute zero

Free Electron Model

- The degeneracy pressure is given by

$$P = -\frac{\partial U}{\partial V} \text{ and } U = \frac{3}{5} N E_F = \frac{3^{5/3}\pi^{4/3}}{5} \frac{\hbar^2}{2m} N^{5/3} V^{-2/3}$$

$$\text{Then } P_{\deg} = \frac{6}{5}\left(\frac{\pi^4}{3}\right)^{1/3} \frac{\hbar^2}{2m}\left(\frac{N}{V}\right)^{5/3}$$

- For a typical metal this is

$$P_{\deg} \approx \frac{\left(1\times10^{-34}\right)^2}{9\times10^{-31}}\left(16\times10^{28}\right)^{5/3} \approx 5\times10^{10}\,Pa$$

$$\text{Recall } 1\,atm = 1\times10^{5}\,Pa$$

- This enormous pressure is counteracted by the Coulomb attraction of the electrons to the positive ions

Free Electron Model

- This degenerate pressure is important in stellar evolution
  - Stars burn by fusion
  - e.g. $^{1}H + {}^{1}H \rightarrow {}^{2}H + e + \nu + 1.44 MeV$
- For stars that are burning (fusion) the gas and radiation pressure supports the star against gravitational compression
- When the star becomes primarily ^{12}C, ^{16}O or heavier the burning (fusion) stops and the star resumes gravitational collapse
  - The only barrier to this is the degeneracy pressure

Free Electron Model

- The gravitational potential energy is calculated as

$$dU_g = -G\frac{mdm}{r} = -G\frac{\left(\frac{4\pi\rho r^3}{3}\right)\left(4\pi\rho r^2 dr\right)}{r}$$

$$dU_g = -\frac{16\pi^2 G\rho^2}{3} r^4 dr$$

and integrating we find

$$U_g = -\frac{16\pi^2 G\rho^2}{3}\int_0^R r^4 dr = -\frac{16\pi^2 G\rho^2}{15} R^5$$

we can rewrite this using $\rho = \frac{N_{nuc} m_p}{V}$ and $V = \frac{4\pi R^3}{3}$

$$U_g = -\left(\frac{3}{5}\right)\left(\frac{4\pi}{3}\right)^{1/3} G\left(N_{nuc} m_p\right)^2 V^{-1/3}$$

Free Electron Model

- Then the gravitation pressure is

$$P_g = -\frac{\partial U_g}{\partial V} = +\frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3} G\left(N_{nuc} m_p\right)^2 V^{-4/3}$$

- Which can be compared to our previous result for the degeneracy pressure

$$P_{\deg} = \frac{6}{5}\left(\frac{\pi^4}{3}\right)^{1/3} \frac{\hbar^2}{2m_e}\left(\frac{N_{nuc}}{2V}\right)^{5/3}$$

$$\text{where } N_e \approx \frac{N_{nuc}}{2}$$

Free Electron Model

- Once fusion ends, gravitational pressure causes the star to collapse until the gravitational pressure and degeneracy pressure balance
- This happens at

$$R = \left(\frac{81\pi^2}{128}\right)^{1/3}\left(\frac{\hbar^2}{G m_e m_p^2}\right) N_{nuc}^{-1/3} \approx 1.15\times 10^{23} km\left(N_{nuc}^{-1/3}\right)$$

- Stars that reach this state are called white dwarfs
  - Most small and medium size stars (like our sun) end up as white dwarfs

Free Electron Model

- White dwarfs have initial masses of M < 3-4 M_{sun} but radii on the order of R_{earth}!
  - We can calculate R using the results of our calculation

$$N \approx \frac{M_{sun}}{m_p} = \frac{2\times10^{30}\,kg}{1.7\times10^{-27}\,kg} = 1.2\times10^{57}$$

$$\text{And } R = \left(\frac{81\pi^2}{128}\right)^{1/3}\left(\frac{\hbar^2}{G m_e m_p^2}\right) N_{nuc}^{-1/3} \approx 1.15\times10^{23}\,km\left(N_{nuc}^{-1/3}\right)$$

$$\text{So } R \approx 1.1\times10^{4}\,km \text{ (about 2x } R_{earth}\text{)}$$

- Or a teaspoon of white dwarf would weigh ~ 5 tons

Free Electron Model

- If M_{star} > ~1.4 M_{sun}, there are larger numbers of electrons with larger energies
- If N_e is large enough, the electrons become relativistic and $E \approx pc$ instead of $p^2/2m$
- This changes the expression for the degeneracy pressure and in fact the gravitational pressure overcomes the degeneracy pressure and collapse continues

Free Electron Model

- This further collapse causes

$$e^- + p \rightarrow n + \nu$$

- This means the star becomes a neutron star and now we calculate the degeneracy pressure using identical neutrons instead of identical electrons
  - For M_{star}~$2M_{sun}$, equilibrium between the gravitational and degeneracy pressure can be reached and R ~ 10 km!

- If M_{star} > several M_{sun}, there will be more neutrons and their energy will become relativistic
  - At this point there is no mechanism to counterbalance the huge gravitational pressure and a black hole is formed

Band Theory

- The free electron model was successful in predicting a number of properties of metals
  - Conduction electrons had relatively long mean free paths l in order to gain agreement with experimental conductivity

$$\sigma = \frac{ne^2 l}{mv_F} = \frac{1}{\rho}$$

  - Electron-nucleus (ion) and electron-electron interactions were ignored
  - The model isn't appropriate for semiconductors or insulators

Band Theory

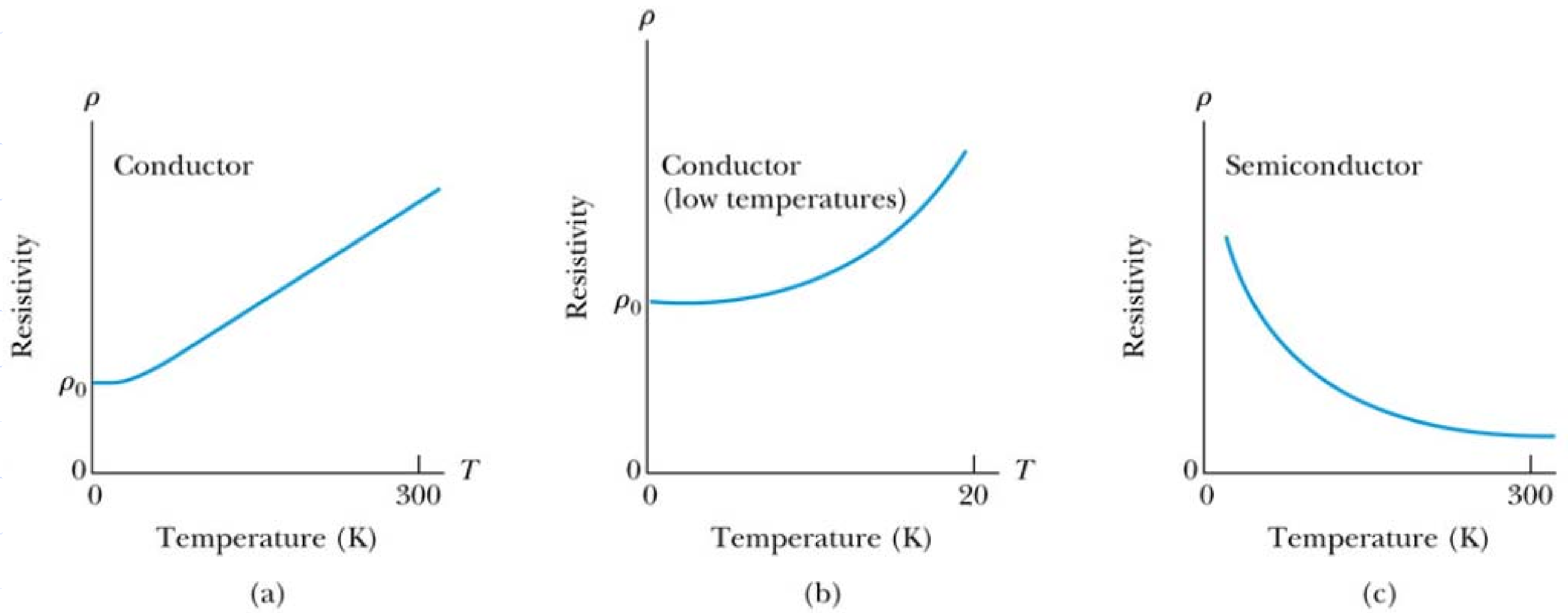
- The free electron model was successful in predicting a number of properties of metals
  - Conduction electrons had to have relatively long mean free paths l in order to gain agreement with experimental conductivity

$$\sigma = \frac{ne^2 l}{m v_F} = \frac{1}{\rho}$$

  - Electron-nucleus (ion) and electron-electron interactions were ignored
  - The model isn't appropriate for semiconductors or insulators

Band Theory

- There are striking differences between conductors and semiconductors



© 2006 Brooks/Cole - Thomson

- Recall R=ρL/A

Band Theory

Table 11.1 Electrical Resistivity and Conductivity of Selected Materials at 293 K

Material	Resistivity ($\Omega \cdot m$)	Conductivity ($(\Omega^{-1} \cdot m^{-1})$)
Metals		
Silver	1.59×10^{-8}	6.29×10^{7}
Copper	1.72×10^{-8}	5.81×10^{7}
Gold	2.44×10^{-8}	4.10×10^{7}
Aluminum	2.82×10^{-8}	3.55×10^{7}
Tungsten	5.6×10^{-8}	1.8×10^{7}
Platinum	1.1×10^{-7}	9.1×10^{6}
Lead	2.2×10^{-7}	4.5×10^{6}
Alloys		
Constantan	4.9×10^{-7}	2.0×10^{6}
Nichrome	1.5×10^{-6}	6.7×10^{5}
Semiconductors		
Carbon	3.5×10^{-5}	2.9×10^{4}
Germanium	0.46	2.2
Silicon	640	1.6×10^{-3}
Insulators		
Wood	10^{8}–10^{11}	10^{-8}–10^{-11}
Rubber	10^{13}	10^{-13}
Amber	5×10^{14}	2×10^{-15}
Glass	10^{10}–10^{14}	10^{-10}–10^{-14}
Quartz (fused)	7.5×10^{17}	1.3×10^{-18}

© 2006 Brooks/Cole - Thomson

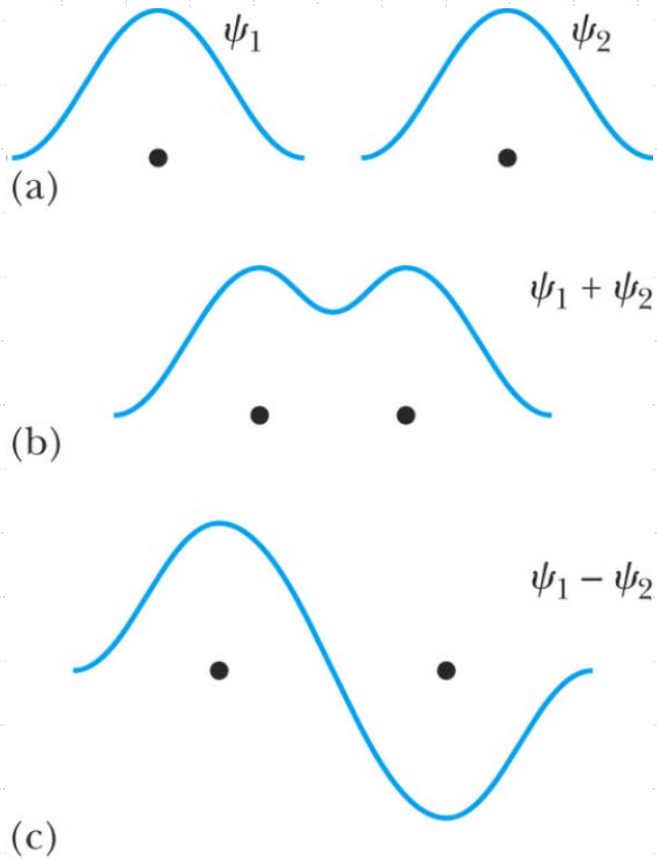
Band Theory

- Band theory incorporates the effects of electrons interacting with the crystal lattice
  - The interaction with a periodic lattice results in energy bands (grouped energy levels) with allowed and forbidden energy regions
- There are two standard approaches to finding the energy levels
  - Bring individual atoms together to form a solid
  - Apply the Schrodinger equation to electrons moving in a periodic potential

Band Theory

- Consider first two atoms (say Na)
  - When the atoms are well separated there is little overlap between the electronic wave functions
    - The energy is two-fold degenerate
  - When the atoms are brought closer together, the electronic wave functions will overlap and we must use wavefunctions with the proper symmetry
    - As the two atoms move closer together the degeneracy is broken

Band Theory

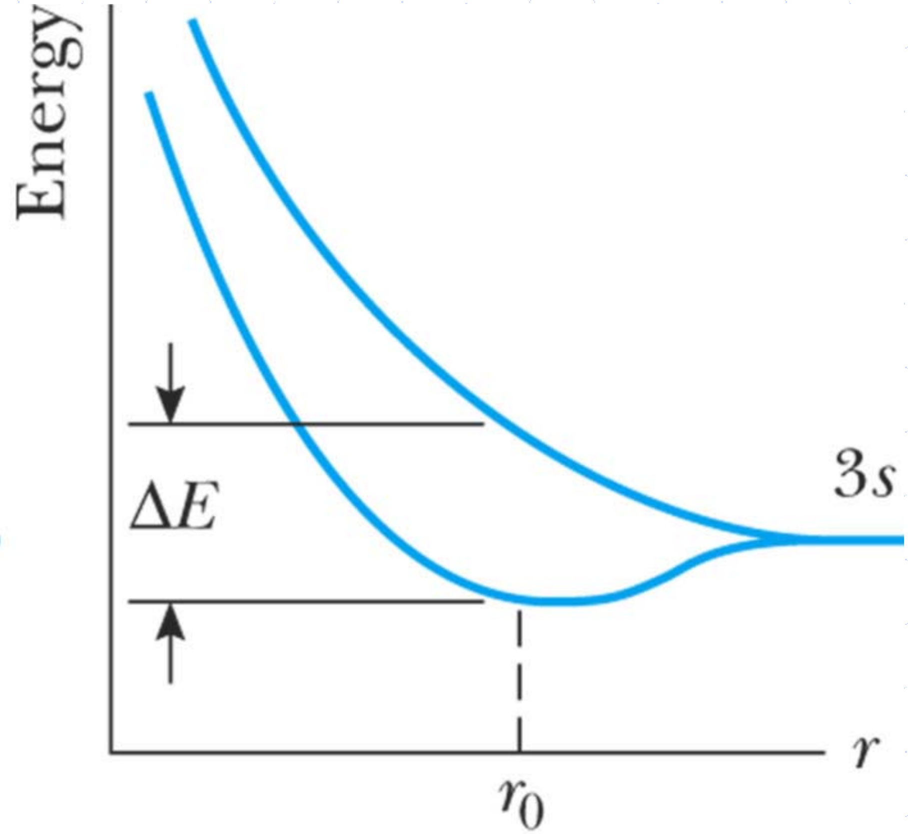


© 2005 Brooks/Cole - Thomson

- The symmetric combination will be lower in energy because the ions exert a greater Coulomb attractive force

Band Theory

- Energy levels of two atoms as they are brought close together



(a)

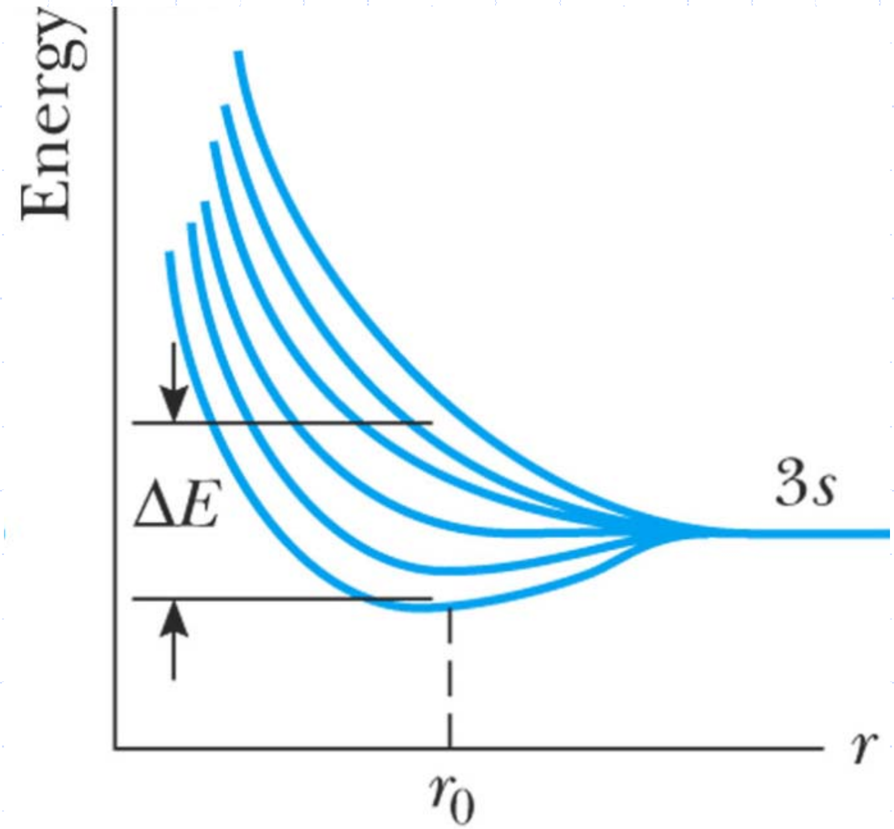
© 2005 Brooks/Cole - Thomson

Band Theory

- Consider next six atoms
  - Again, as the atoms are brought closer together there will be symmetric and antisymmetric wavefunctions
  - But there will also be wavefunctions of mixed symmetry and the corresponding energies lie between the two
  - Note the energy increases as the separation R becomes smaller because of the Coulomb repulsion of the ions

Band Theory

- Energy levels of six atoms as they are brought close together

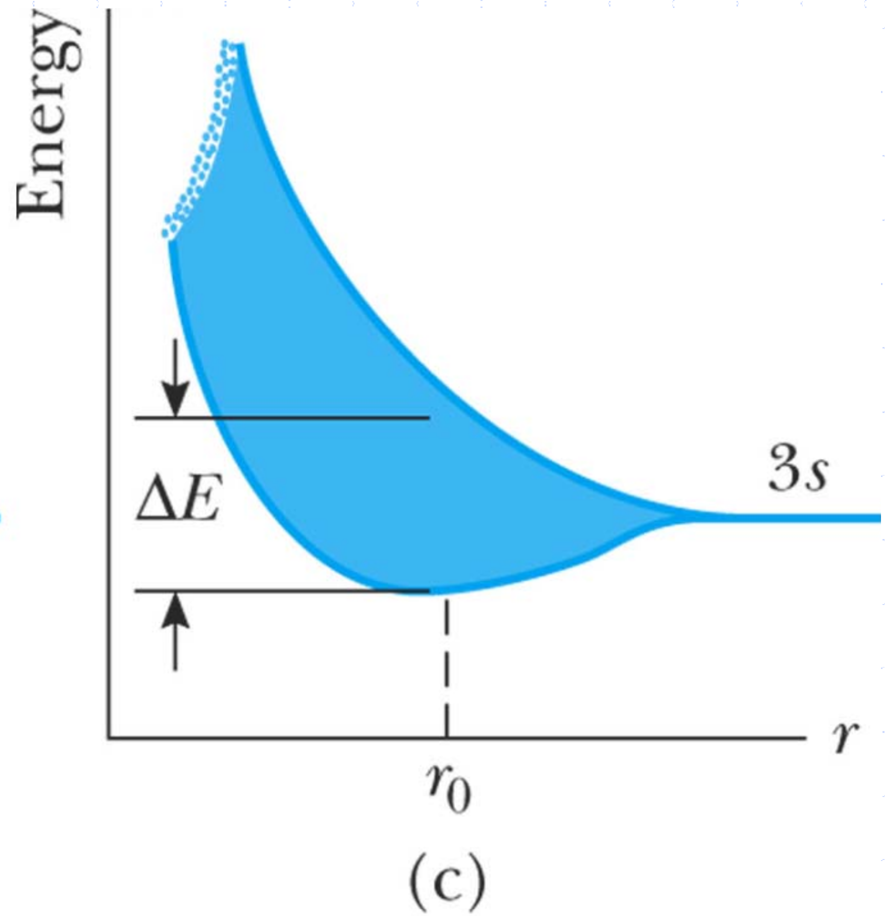


(b)

© 2005 Brooks/Cole - Thomson

Band Theory

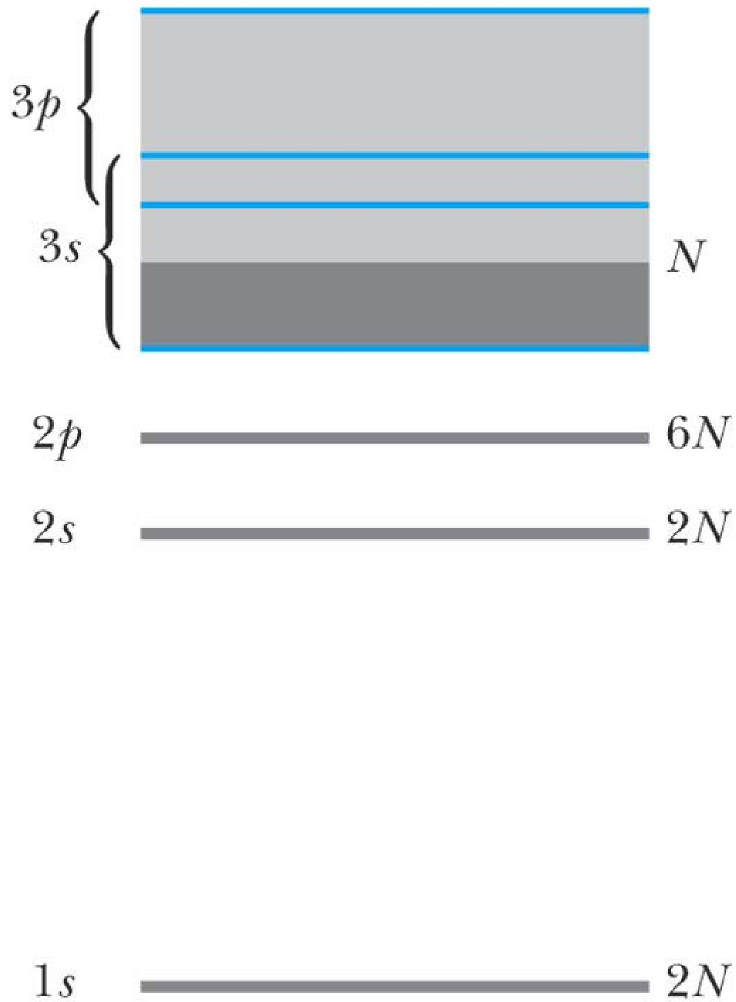
- Energy levels of N atoms as they are brought close together – band formation



© 2005 Brooks/Cole - Thomson

Band Theory

- For higher (or lower) energy levels an energy gap may or may not exist depending on the type of atom, type of bonding, lattice spacing, and lattice structure



© 2005 Brooks/Cole - Thomson