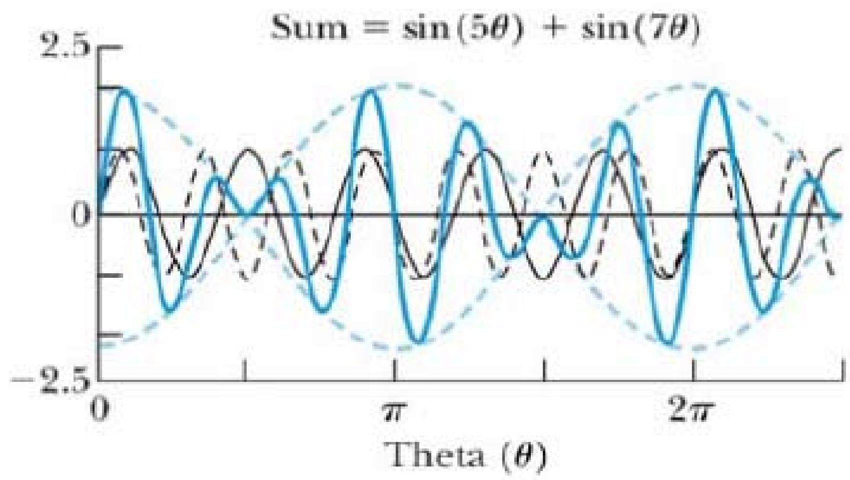
Waves

- The first term is the wave and the second term is the envelope

$$\Psi(x,t) = 2A\cos(k_{av}x - \omega_{av}t)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$



Waves

- What is the size in space and time of the wave packet?
- For the moment, let's define the size of the localized wave to be $\Delta x = x_2 - x_1$ where points 1 and 2 are points where the envelope is zero

$$\frac{\Delta k}{2} x_2 - \frac{\Delta k}{2} x_1 = \pi$$

$$\Delta k \Delta x = 2\pi$$

and similarly

$$\frac{\Delta t}{2} \omega_2 - \frac{\Delta t}{2} \omega_1 = \pi$$

$$\Delta \omega \Delta t = 2\pi$$

Waves

- We learned a lot from just summing two waves, but to localize the particle and remove the auxillary waves we must sum/integrate over many waves
- This can be written as a Fourier integral

$$\Psi(x,t) = \int_{-\infty}^{\infty} dk A(k) e^{i(kx-\omega t)}$$

- The above form is equivalent to the one used in your book

$$\Psi(x,t) = \int_{-\infty}^{\infty} dk A(k) \cos(kx-\omega t)$$

Waves

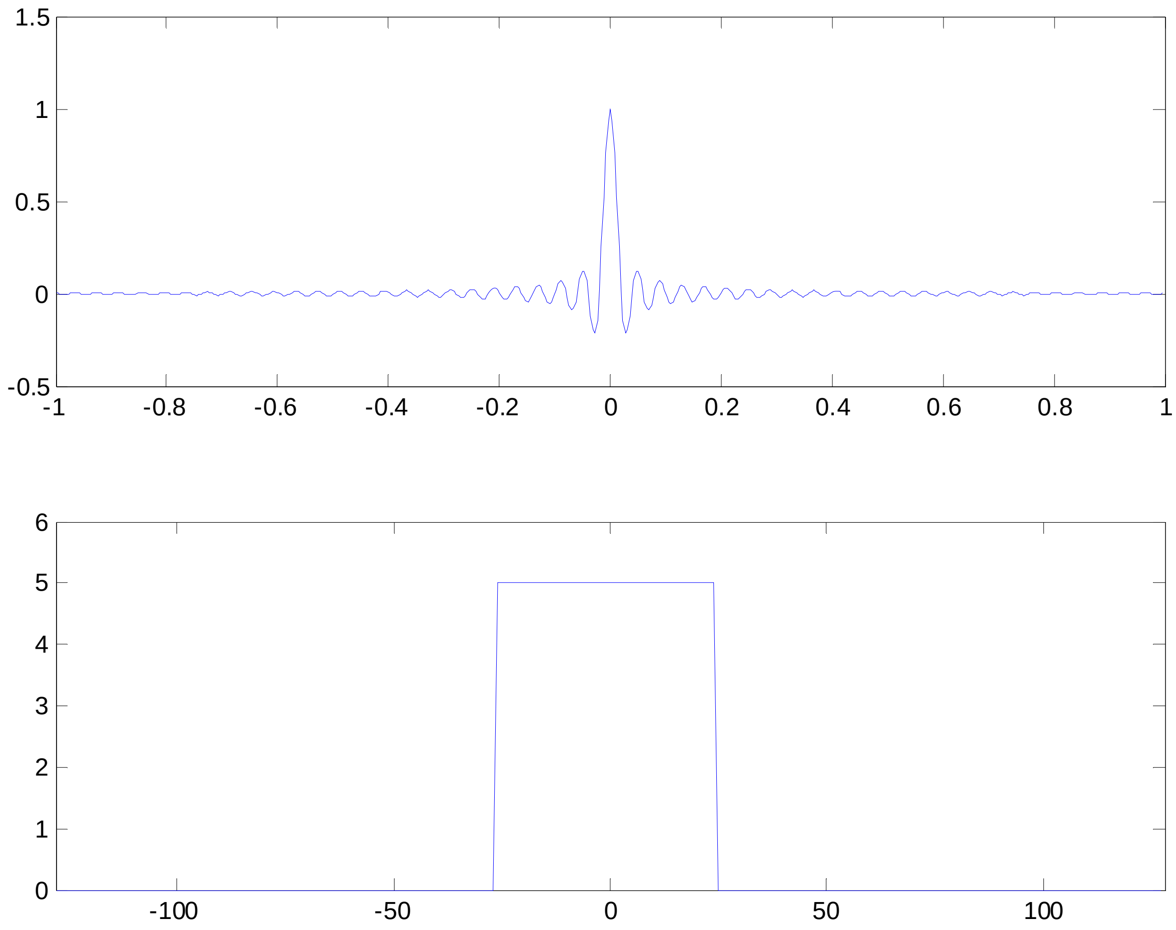
- Problem 5.31

$$\text{let } \tilde{A}(k) = A_0 \text{ for } k = k_0 \mp \Delta k$$
$$= 0 \text{ otherwise}$$

Sketch the envelope term, the oscillating term, and $|\Psi(x,0)|^2$

Defining Δx to be the FWHM, find $\Delta x \Delta k$

Waves



Waves

- These $\Delta x \Delta k$ relations mean
  - If you want to localize the wave to a small position Δx you need a large range of wave numbers Δk
  - If you want to localize the wave to a small time domain Δt you need a large range of frequencies (bandwidth) $\Delta \omega$

Waves

- Consider the wave packet at t=0

$$\Psi(x,0) = \int_{-\infty}^{\infty} dk A(k) e^{ikx}$$

then $A(k)$ is the Fourier transform of $\Psi(x,0)$

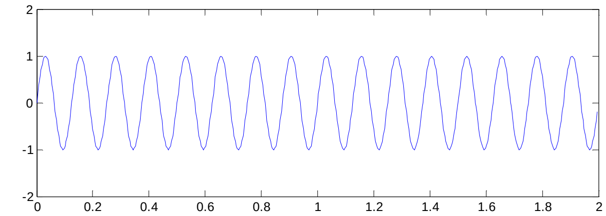
$$A(k) = \int dx \Psi(x,0) e^{-ikx}$$

- The Fourier transform just transforms one function into another
  - Position space is transformed into wave number space
  - Time space is transformed into frequency space

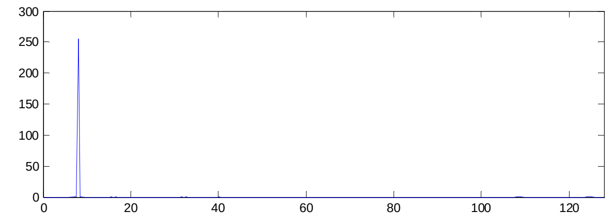
Waves

➢ Some simple examples

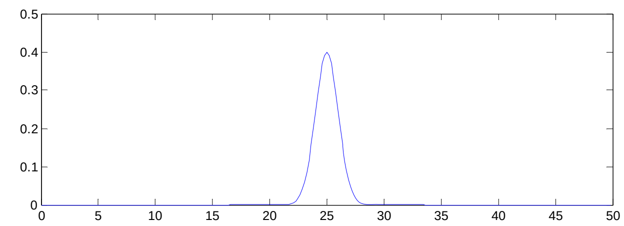
- Space (cosine)



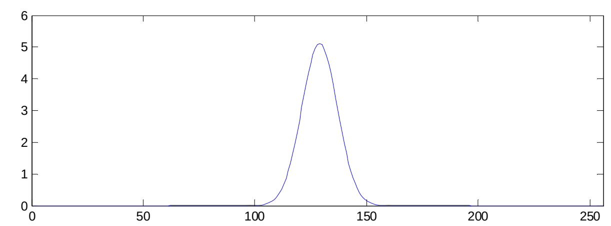
- Wave number (delta function)



- Space (gaussian)



- Wave number (gaussian)



Waves

- A particularly useful wave packet is the gaussian wave packet
  - Both the wave packet and the Fourier transform are gaussian functions
  - It's useful because the math is easier

$$\Psi(x,0) = \int_{-\infty}^{\infty} dk A(k) e^{ikx}$$

$$\text{with } A(k) = e^{-\alpha(k-k_0)^2/2}$$

  - A(k) is centered at k_0 and falls away rapidly from the center

Waves

- Doing the integral (by first changing variables to $q=k-k_0$) gives

$$\Psi(x,0)=\int_{-\infty}^{\infty} dk A(k) e^{ikx}$$

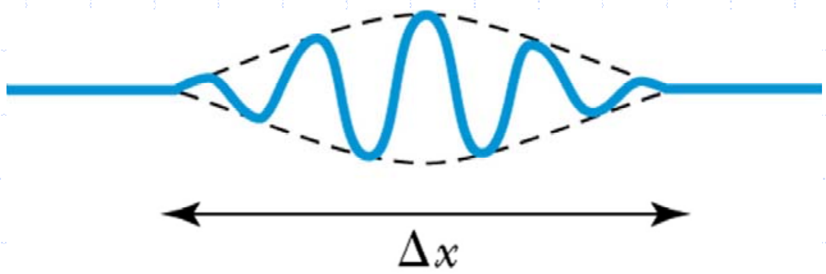
$$\text{with } A(k)=e^{-\alpha(k-k_0)^2/2} \text{ gives}$$

$$\Psi(x,0)=\sqrt{\frac{2\pi}{\alpha}}e^{ik_0x}e^{-\frac{x^2}{2\alpha}}$$

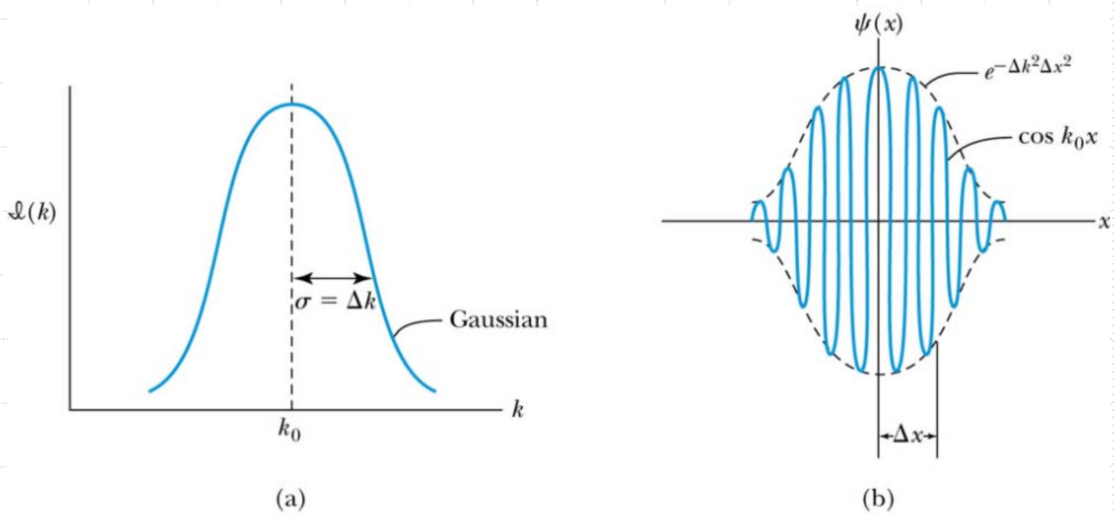
- The factors represent an oscillating wave with wave number k_0 and a modulating envelope
  - The wave function $\Psi(x,0)$ and its Fourier transform $A(k)$ are both gaussians

Waves

- An idealized wave packet



- A gaussian wave packet



Waves

- We can define the width Δx of the gaussian to be the width between which the gaussian function is reduced to $1/\sqrt{e}$

$$e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

$$-\frac{1}{2} = -\frac{x^2}{2\alpha} \text{ or } \Delta \mathrm{x} = 2\sqrt{\alpha}$$

and doing the same for the k - space gaussian

$$-\frac{1}{2} = -\alpha\frac{(\Delta k)^2}{2} \text{ or } \Delta \mathrm{k} = \frac{2}{\sqrt{\alpha}}$$

then $\Delta \mathrm{x} \Delta \mathrm{k} = 4$

Waves

- Slightly different definitions of the width would give slightly different results
  - Still, the more localized the wave packet, the larger the spread of associated wave numbers
- A general result of Fourier integrals is

$$\Delta x \Delta k > \frac{1}{2} \text{ and since } p = \hbar k$$

$$\Delta x \Delta p > \frac{\hbar}{2}$$

- This is the Heisenberg uncertainty principle
  - This is a QM result arising from a property of Fourier transforms and de Broglie waves

Waves

- Time evolution of the wave packet

The wave function at t = 0 is

$$\Psi(x,0) = \int_{-\infty}^{\infty} dk A(k) e^{ikx}$$

The wave function at later times is

$$\Psi(x,t) = \int_{-\infty}^{\infty} dk A(k) e^{i(kx-\omega t)}$$

One can show (and we won't)

$$\Delta x(t) = \sqrt{(\Delta x(0))^2 + \left(\frac{\hbar t}{2m\Delta x(0)}\right)^2}$$

This means the wave is dispersing with time

Waves

- Consider an electron confined to 0.1nm with k_0=0 (at rest). When will the wave packet expand to √2 times its initial size

$$\Delta x^2(0) = \left(\frac{\hbar t}{2m\Delta x(0)} \right)^2$$

$$t = \frac{(\Delta x)^2 2m}{\hbar}$$

$$t = \frac{(0.1nm)^2 (2)(0.511 \times 10^6 eV)}{(197.33eVnm)(3 \times 10^8 m/s)\left(10^9 \frac{nm}{m}\right)}$$

$$t = 1.7 \times 10^{-16} s$$