Waves

- Electrons (and other particles) exhibit both particle and wave properties
  - Particles are described by classical or relativistic mechanics
  - Waves will be described by wave mechanics = quantum mechanics

Waves

- Something disturbing we'll have to address

deBroglie relations $p = \frac{h}{\lambda}$ and $E = hf$

wave velocity $v = \lambda f$

so $v = \frac{h}{p}\frac{E}{h} = \frac{E}{p} = \frac{mV^2/2}{mV} = \frac{V}{2}$

- The wave and particle seem to be moving at different velocities

Waves

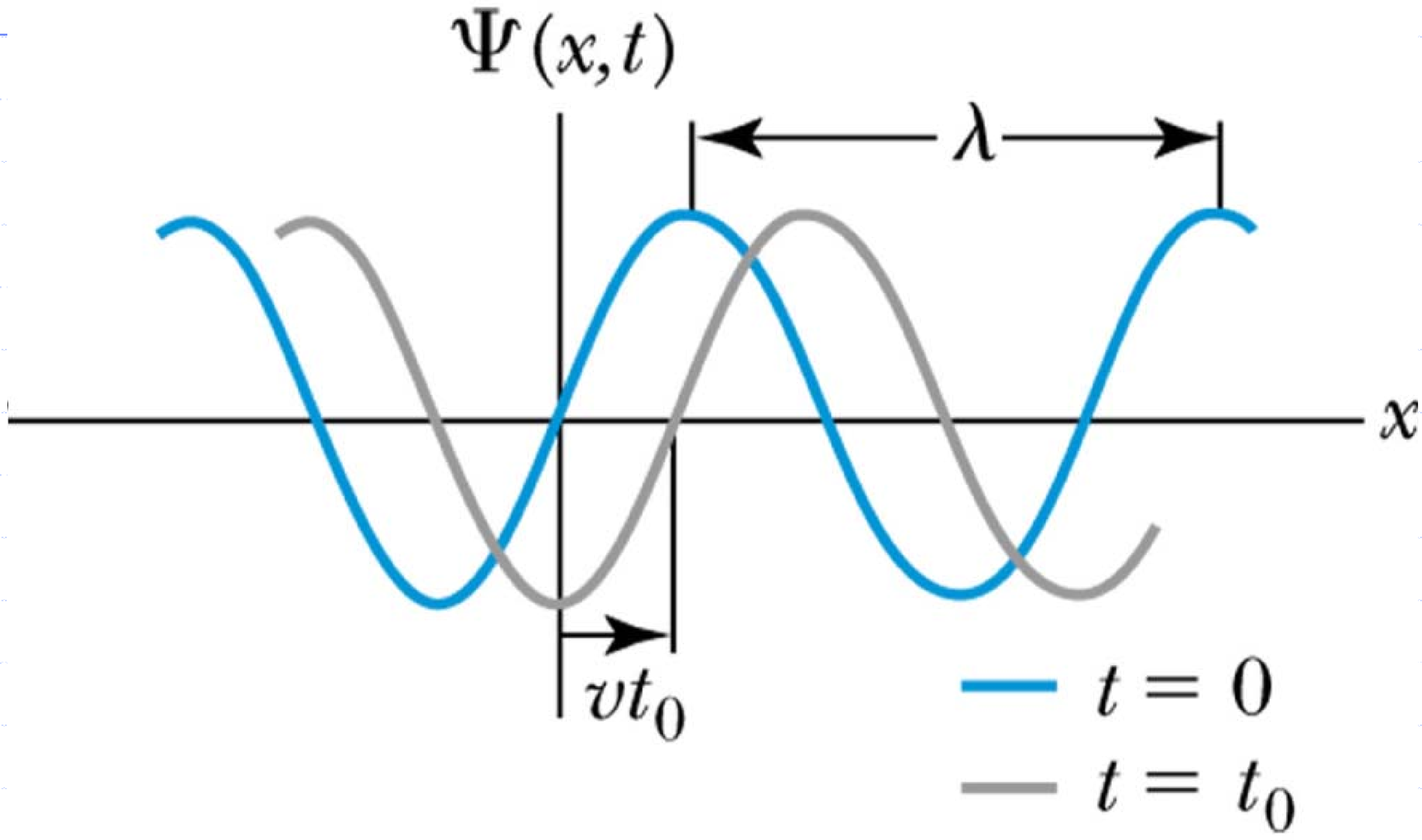
- We'll use Ψ(x,t) to represent the wave function of a particle (electron)
- In lecture 1, we wrote down the classical wave equation that describes a propagating disturbance

$$\frac{d^2\Psi}{dx^2} = \frac{1}{v^2}\frac{d^2\Psi}{dt^2}$$

- A solution to the classical wave equation is

$$\Psi(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

Waves



Waves

$$\frac{d\Psi}{dx} = A\frac{2\pi}{\lambda}\cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$$\frac{d^2\Psi}{dx^2} = -A\left(\frac{2\pi}{\lambda}\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$$\frac{d\Psi}{dt} = -vA\frac{2\pi}{\lambda}\cos\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$$\frac{d^2\Psi}{dt^2} = -Av^2\left(\frac{2\pi}{\lambda}\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$$\text{so } \frac{d^2\Psi}{dx^2} = \frac{1}{v^2}\frac{d^2\Psi}{dt^2} \text{ is satisfied}$$

Waves

- We normally write

$$\Psi(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

as

$$\Psi(x,t) = A\sin(kx - \omega t)$$

where

$$k = \frac{2\pi}{\lambda} \text{ and } \omega = 2\pi f = \frac{2\pi}{T}$$

Waves

- More generally we write

$$A\sin(kx-\omega t+\phi)$$

- Thus for φ=π/2 we have

$$A\cos(kx-\omega t)$$

- The phase velocity is defined as the velocity of a point on wave with a given phase (e.g the velocity of the peak of the wave)

$$v_{phase}=\lambda f=\frac{\lambda}{T}=\frac{\omega}{k}$$

Waves

- The general harmonic wave can also be written as

$$A\sin(kx - \omega t + \phi) = B\sin(kx - \omega t) + C\cos(kx - \omega t)$$

$$\text{where B} = \cos\phi \text{ and C} = \sin\phi$$

Waves

Using Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we can write

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Then other solutions to the wave equation are

$$\mathrm{Im}\{A\exp[i(kx - \omega t + \phi)]\} \text{ or } \mathrm{Re}\{\ldots\}$$

$$\frac{1}{2i}A\exp[i(kx - \omega t + \phi)] + \textit{complex conjugate}\ (c.c.)$$

Waves

Notation

We often write these solutions without the Re, Im, $\frac{1}{2}$, or $\frac{1}{2\mathrm{i}}$

$\mathrm{Im}\{A\exp[i(kx-\omega t+\phi)]\}$ becomes just

$A\exp[i(kx-\omega t+\phi)]$

and furthermore

$A\exp[i(kx-\omega t+\phi)] = A\exp[i\phi]\exp[i(kx-\omega t)]$

can be written

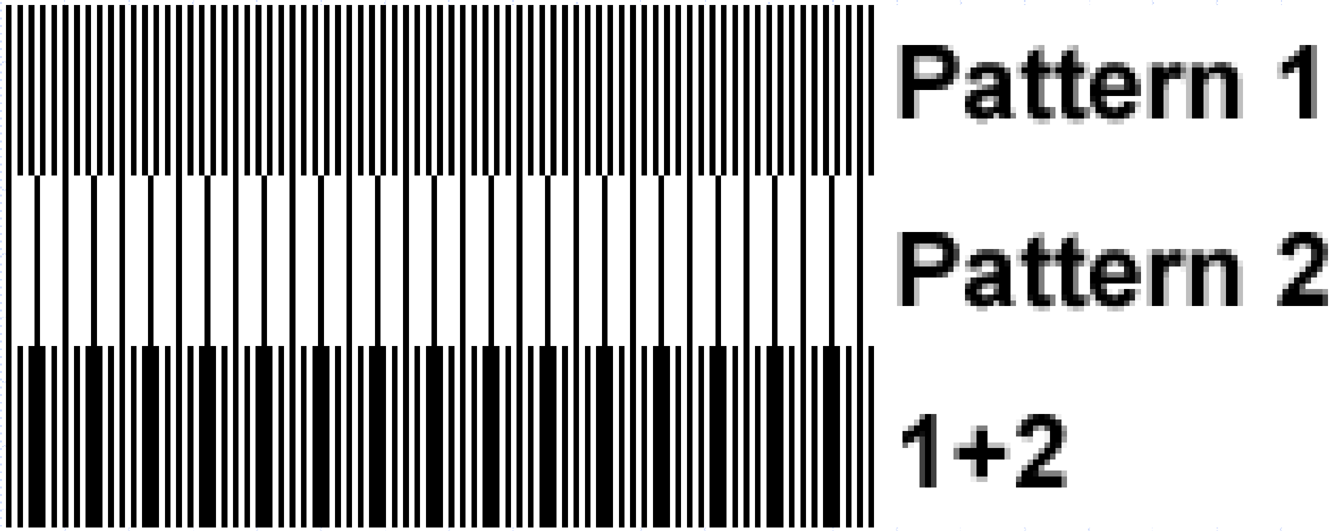
$\tilde{A}\exp[i(kx-\omega t)]$ where $\tilde{A}$ is complex

Again, the ~ is usually understood so the solution becomes

$\mathrm{A}\exp[i(kx-\omega t)]$

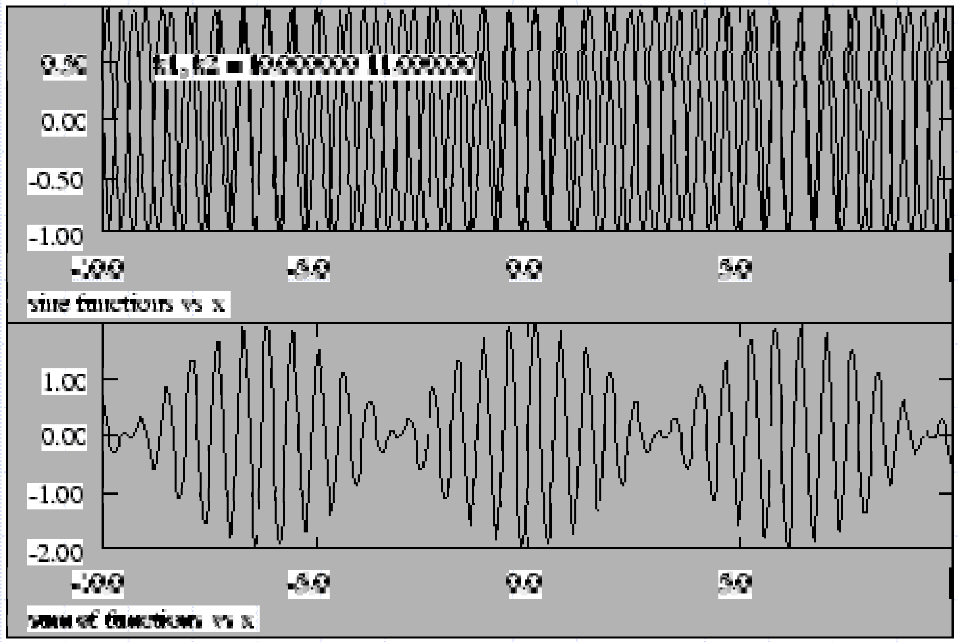
Waves

- How can we represent a (localized) particle by a (non-localized) wave?
  - Moire pattern



Waves

- We can form a wave packet by summing waves of different wavelengths or frequencies



- Summing over an infinitely large number of waves between a range of k values will eliminate the auxiliary wave packets

Waves

- Let's start by adding two waves together

$$\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t)$$

$$\Psi(x,t) = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t)$$

$$\text{using } \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\Psi(x,t) = 2A\cos\left(k_{av}x - \omega_{av}t\right)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

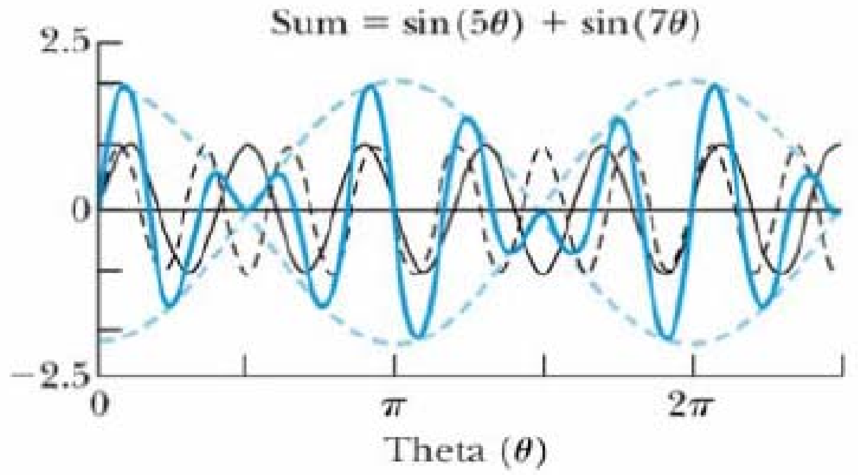
$$\text{where } k_{av} = \frac{k_1 + k_2}{2} \text{ and } \Delta k = k_1 - k_2$$

and similarly for ω

Waves

- The first term is the wave and the second term is the envelope

$$\Psi(x,t) = 2A\cos(k_{av}x - \omega_{av}t)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$



Waves

- The combined wave has phase velocity

$$v_{phase} = \frac{\omega_{av}}{k_{av}}$$

- The group (envelope) has group velocity

$$v_{group} = \frac{\Delta\omega}{\Delta k}$$

- More generally the group velocity is

$$v_{group} = \frac{d\omega(k)}{dk}$$

Waves

- Recall our problem with the wave velocity being slower than the particle velocity
- Using the group velocity we find

$$v_{group} = \frac{d\omega}{dk} = \frac{d\hbar\omega}{d\hbar k} = \frac{dE}{dp}$$

$$E = \frac{p^2}{2m}$$

$$v_{group} = \frac{p}{m} = V$$

- Problem solved

Waves

- The relation between ω and k is called the dispersion relation
- Examples
  - Photon in vacuum

$$f = \frac{c}{\lambda}$$

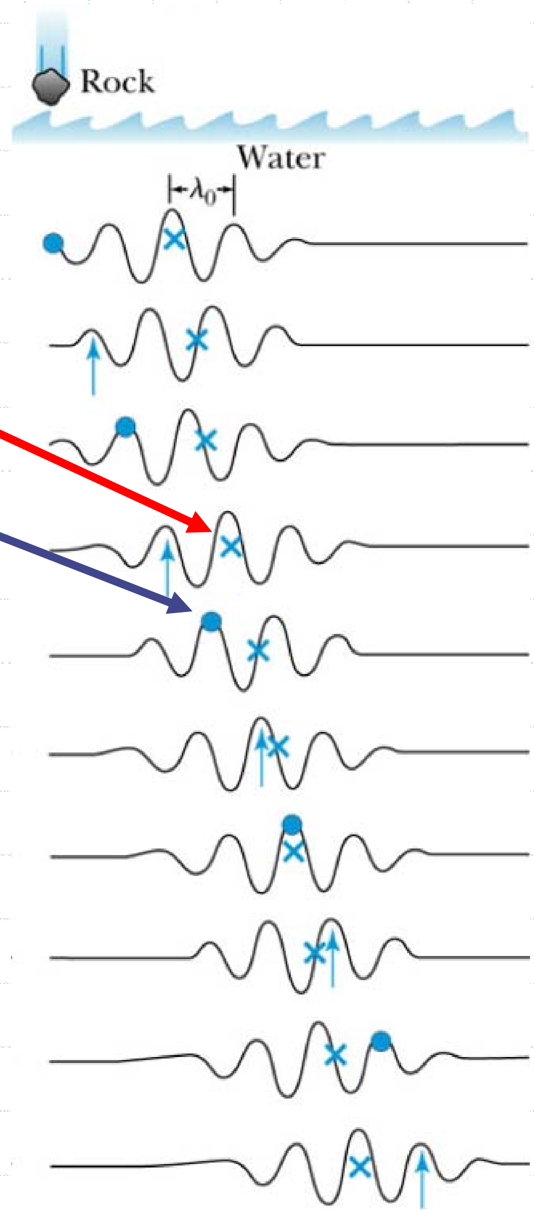
$$2\pi f = \frac{2\pi c}{\lambda}$$

$$\omega = kc$$

$$\frac{d\omega}{dk} = c$$

Waves

- Phase and group velocity



Waves

- Examples
  - Photon in a medium

$$f = \frac{c}{n(\lambda)\lambda}$$

$$2\pi f = \frac{2\pi c}{n(\lambda)\lambda}$$

$$\omega = \frac{kc}{n(k)}$$

$$\frac{d\omega}{dk} = \frac{c}{n(k)} + kc\frac{d}{dk}\frac{1}{n(k)}$$

Waves

- Examples
  - de Broglie waves

$$E = hf = \hbar\omega$$

$$E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m}$$

$$\omega = \frac{\hbar k^2}{2m}$$

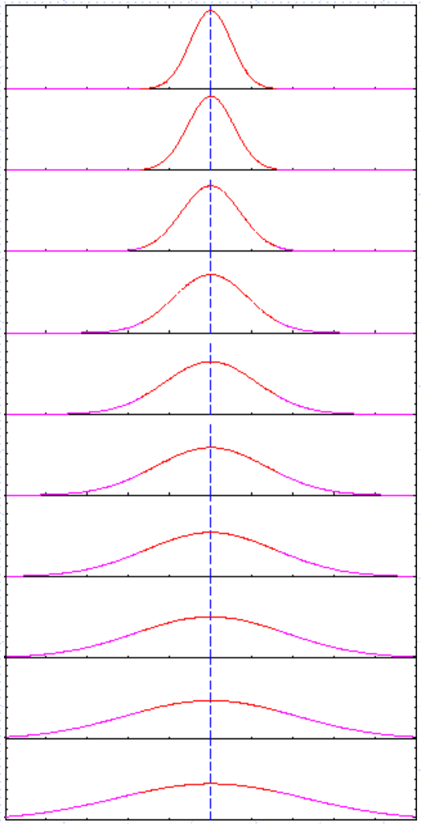
$$\frac{d\omega}{dk} = \frac{\hbar k}{m}$$

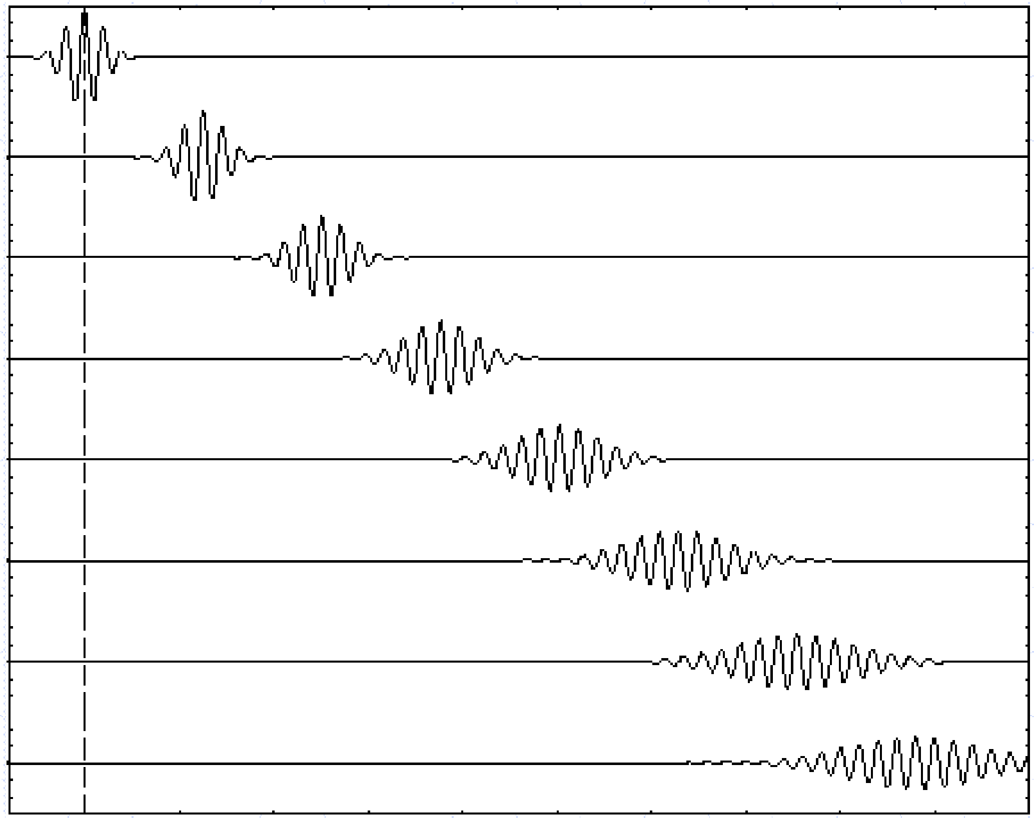
Waves

- Because the group velocity = $d\omega/dk$ depends on k, the wave packet will disperse with time
  - This is because each of the waves is moving at a slightly different velocity
  - Just as white light will be dispersed as it travels through a prism
- That's why $\omega(k)$ is called a dispersion relation

Waves

- Spreading of a wave packet with time





- Does this mean matter waves disperse with time?