Cross Section

- Much of what we have learned about the internal structure of matter comes from scattering experiments (i.e. a microscope)
  - Beam = electron, pion, proton, antiproton, alpha, photon, neutrino, strange particles, nuclei, …
  - Target = electron, proton, alpha, photon, nuclei, …
- Fixed target experiments

- Colliding beam experiments

Cross Section

- Imagine shooting bullets at a circular target behind a curtain
- The bullets are spread uniformly

$$R = I\sigma$$

where

R is the hit frequency (bullets/sec)

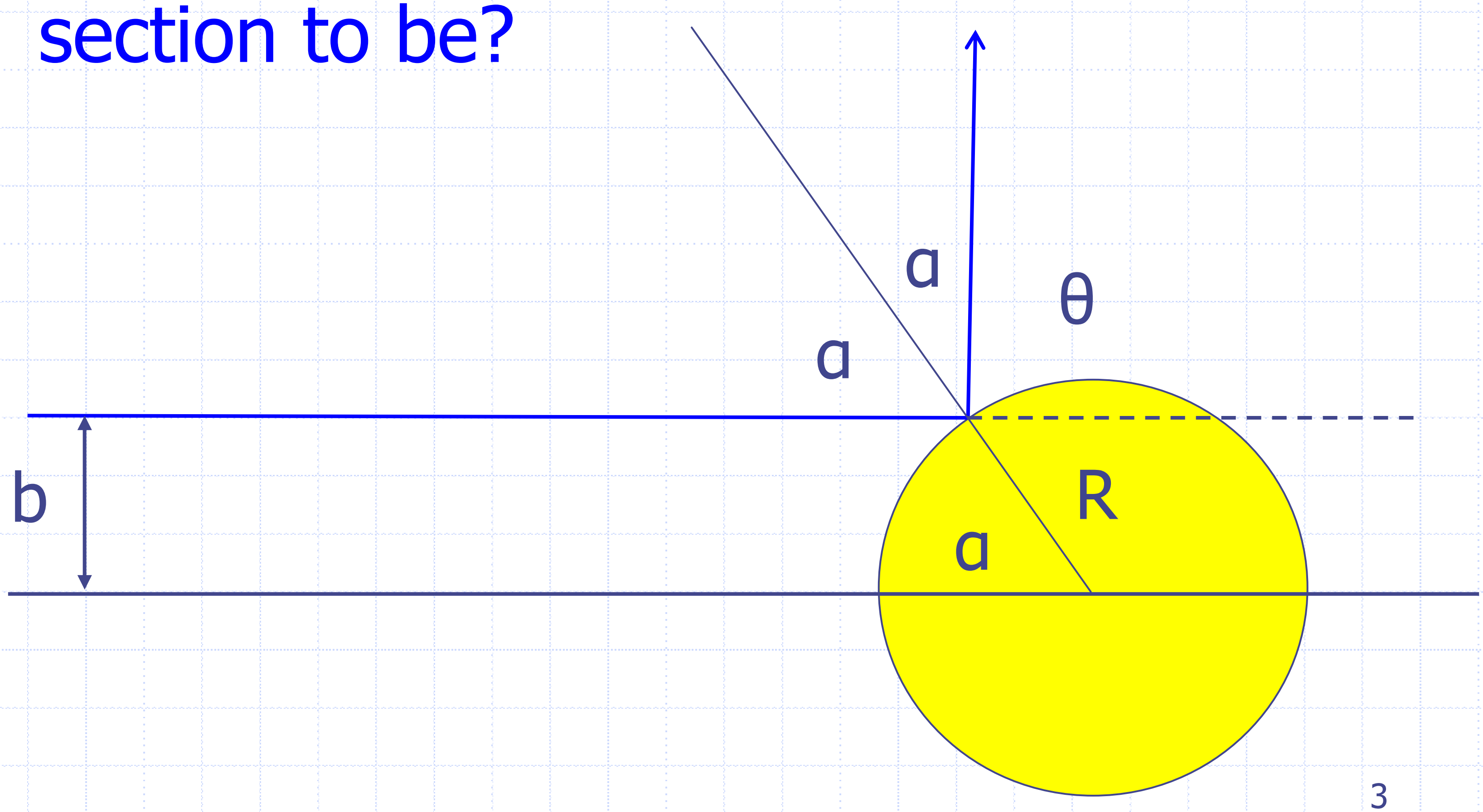
I is the incoming bullet intensity (bullets/m^2/sec)

σ is the cross section (m^2)

- The larger the target (σ) the more hits (scatters)

Cross Section

- Consider scattering from a hard sphere
- What would you expect the cross section to be?



Cross Section

- From the figure we see

$$b = R\sin\alpha \text{ and } 2\alpha + \theta = \pi$$

$$\text{then } \sin\alpha = \sin(\pi/2 - \theta/2) = \cos(\theta/2)$$

$$\text{and } b = R\cos(\theta/2)$$

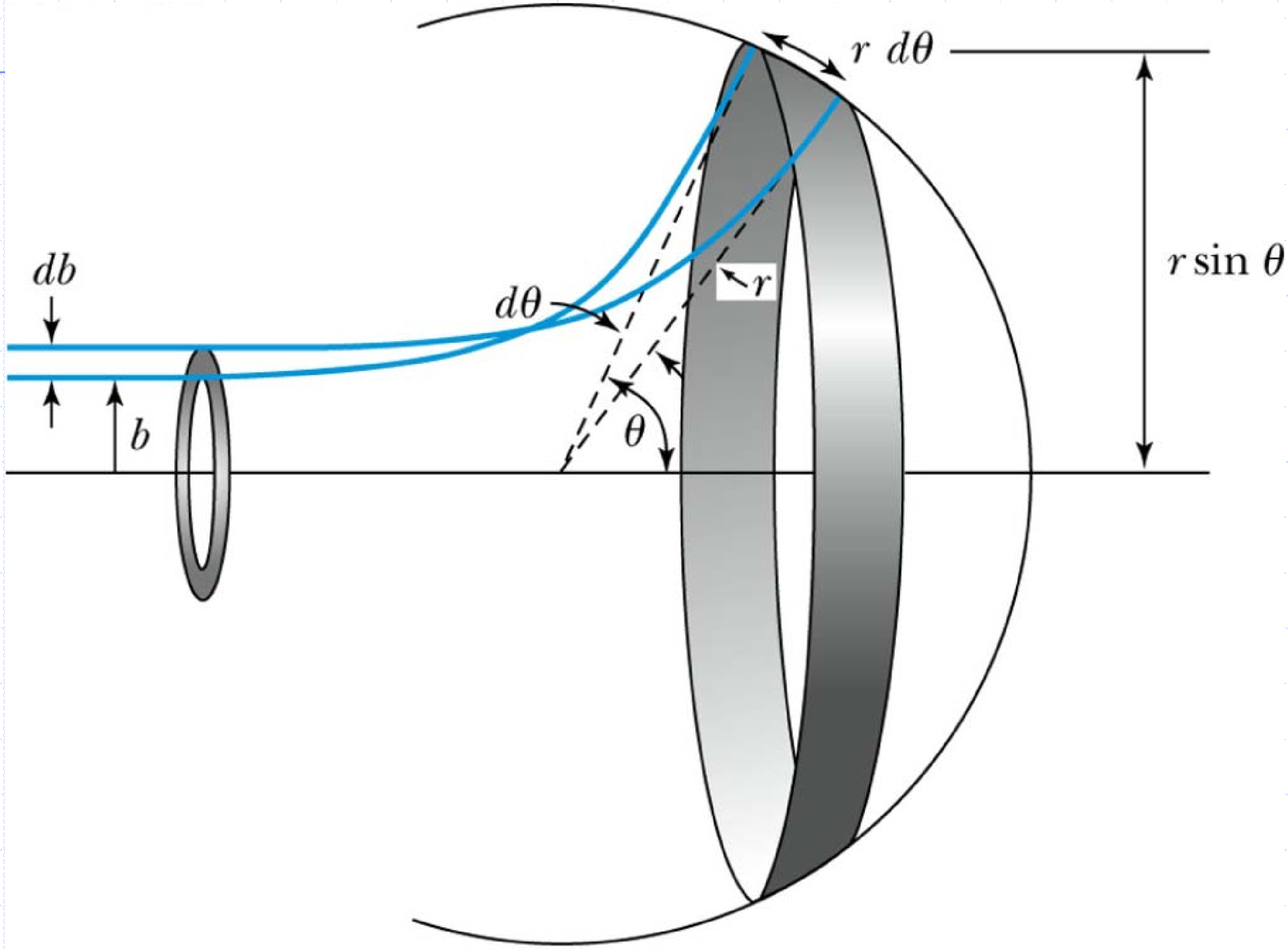
$$\text{or } \theta = 2\cos^{-1}(b/R)$$

- This is the relation between b and θ for hard sphere scattering

Cross Section

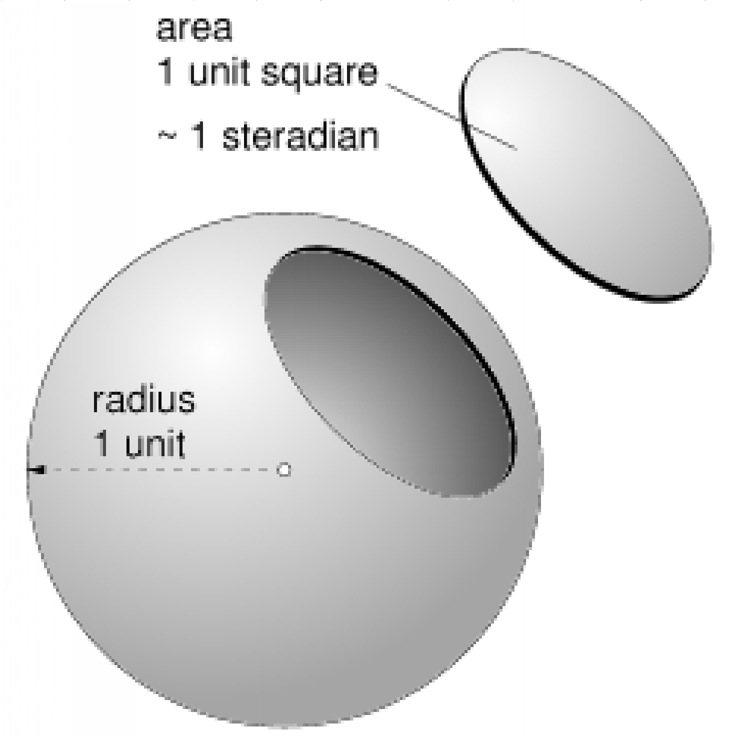
- If a particle arrives with an impact parameter between b and b+db, it will emerge with a scattering angle between θ and θ+dθ
- If a particle arrives within an area of dσ, it will emerge into a solid angle dΩ

Cross Section



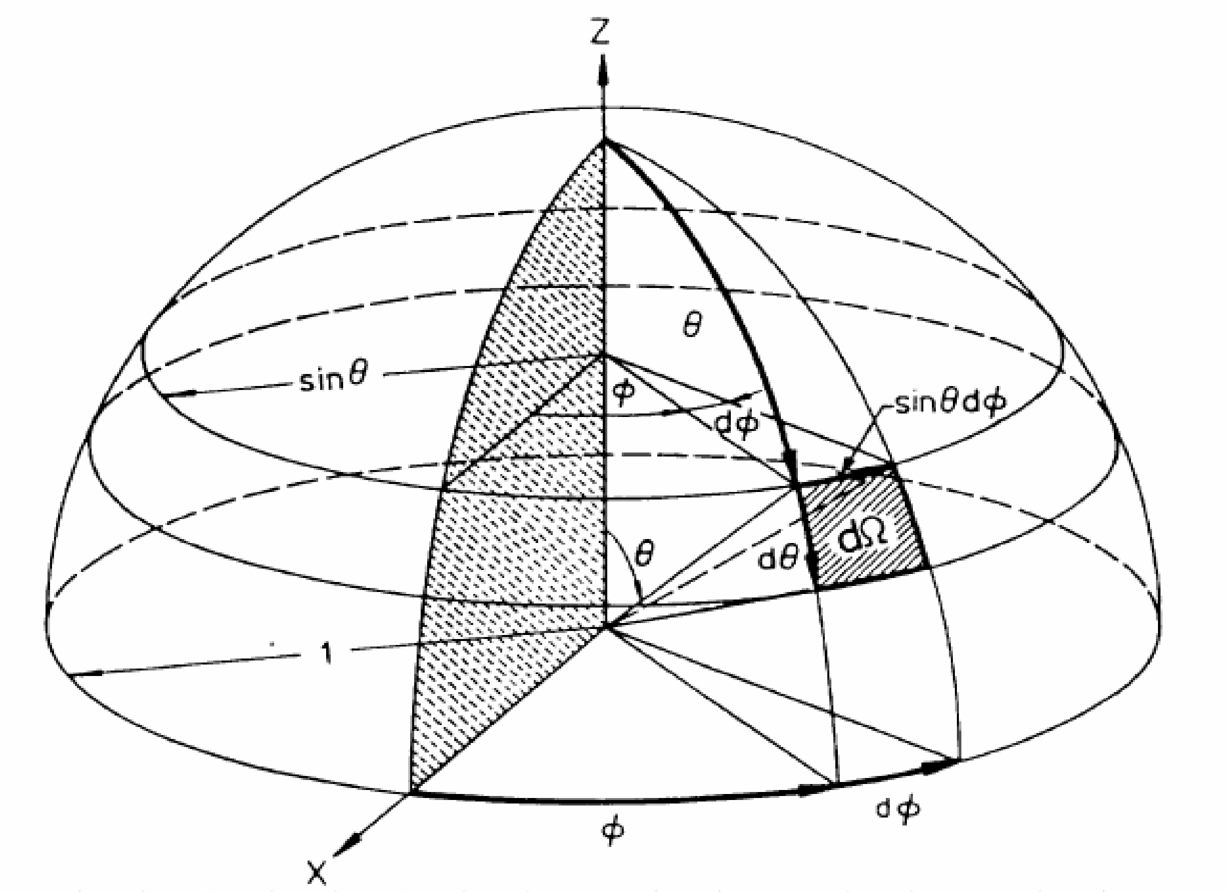
Cross Section

- $\Omega = A/r^2$
- So a sphere has 4π steradians (sr)



Cross Section

- $d\Omega = dA/r^2 = \sin\theta d\theta d\varphi$



Cross Section

- We have

$$d\sigma = \frac{d\sigma}{d\Omega} d\Omega$$

$$\text{where } d\sigma = b\,db\,d\phi$$

$$\text{and} \quad d\Omega = \sin\theta\, d\theta\, d\phi$$

- And the proportionality constant dσ/dΩ is called the differential cross section

Cross Section

- Then we have

$$\frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta} \frac{db}{d\theta} \right|$$

- And for the hard sphere example

$$\frac{db}{d\theta} = -\frac{R}{2}\sin\frac{\theta}{2}$$

$$\frac{d\sigma}{d\Omega} = \frac{Rb\sin\frac{\theta}{2}}{2\sin\theta} = \frac{R^2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{2\sin\theta} = \frac{R^2}{4}$$

Cross Section

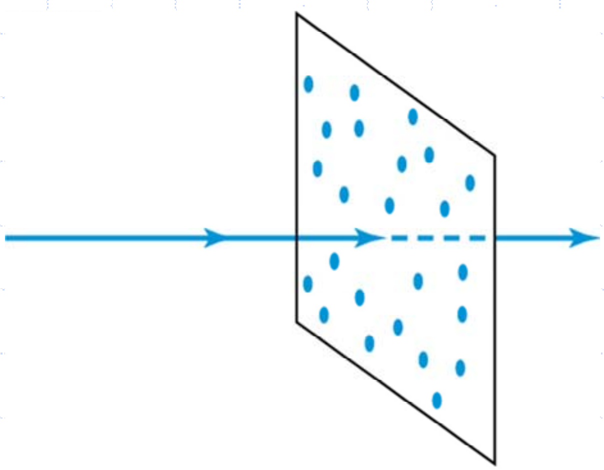
- Finally

$$\sigma = \int d\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int \frac{R^2}{4} d\Omega = \pi R^2$$

- This is just as we expect
- The cross section formalism developed here is the same for any type of scattering (Coulomb, nuclear, …)

Cross Section

- The units of cross section are barns
  - 1 barn (b) = $10^{-28} m^2$
  - The units are area. One can think of the cross section as the effective target area for collisions. We sometimes take $\sigma = \pi r^2$

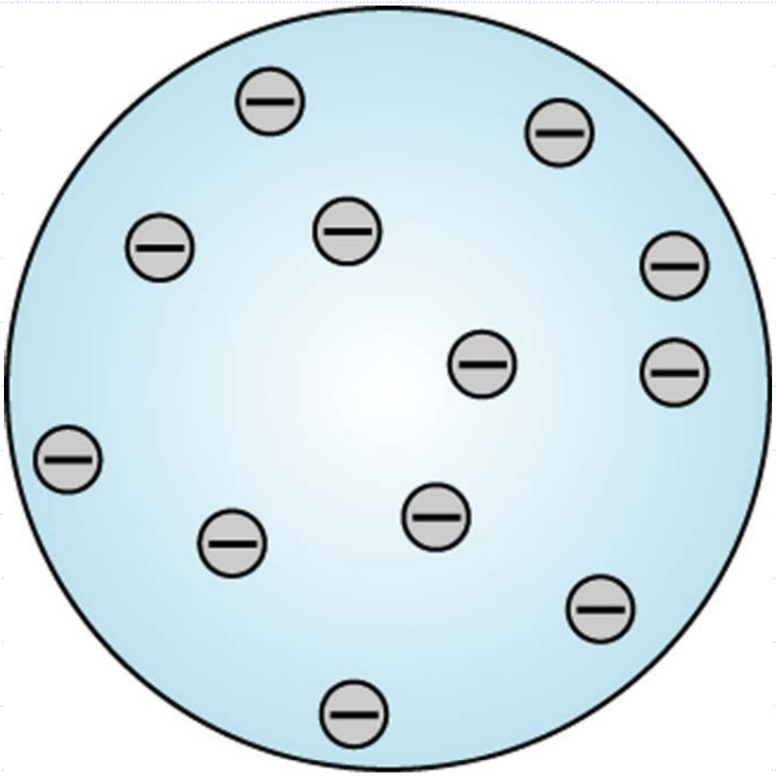


Cross Section

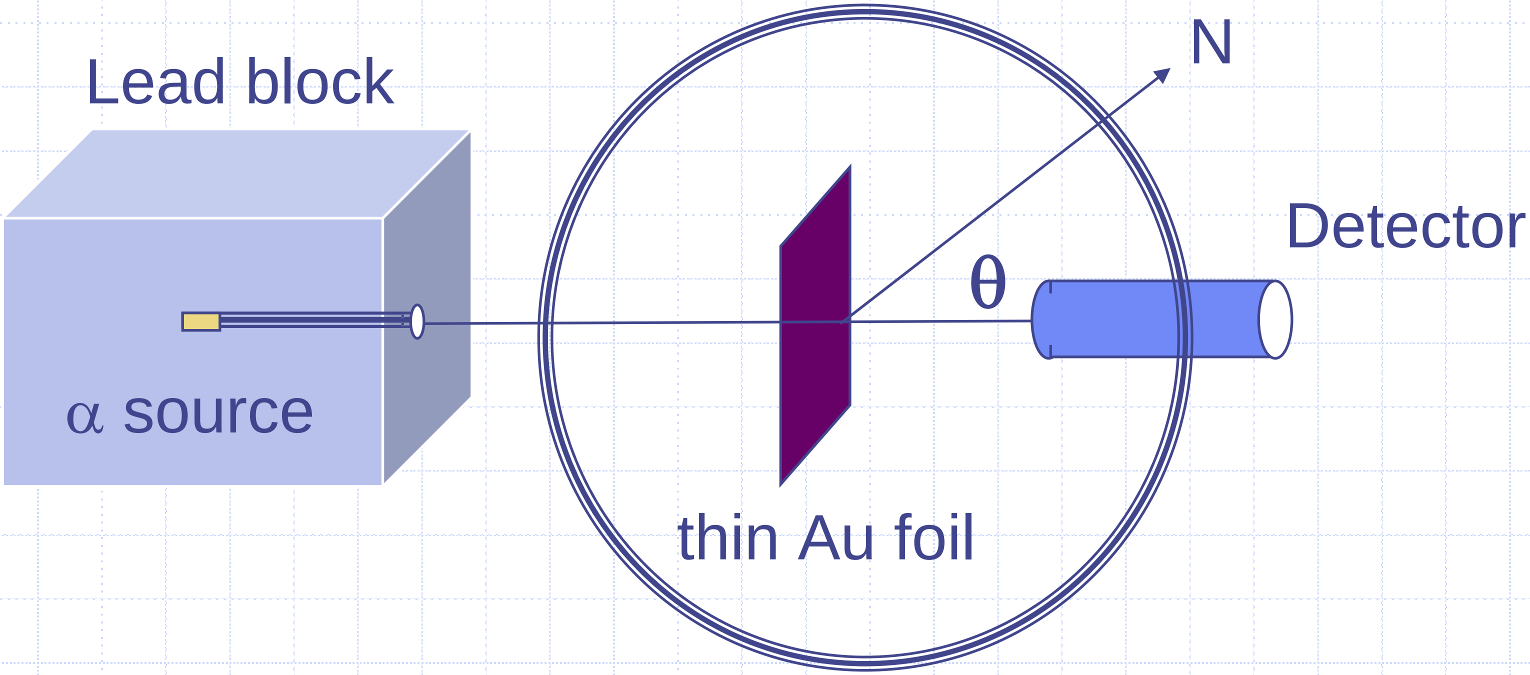
- One can find the scattering rate by
  - $R = I_0 N_T \sigma$
    - number/s = (number/s) (number nuclei/cm^2) (cm^2)
  - $N_T = N_{Av} \rho l / At$
    - nuclei/cm^2 = (Avagadro's number) (density) (length) / (atomic weight in g)
- And the scattering rate into $d\Omega$ is
  - $R(d\Omega) = I_0 N_T d\sigma/d\Omega$

Thomson's Plum Pudding Model

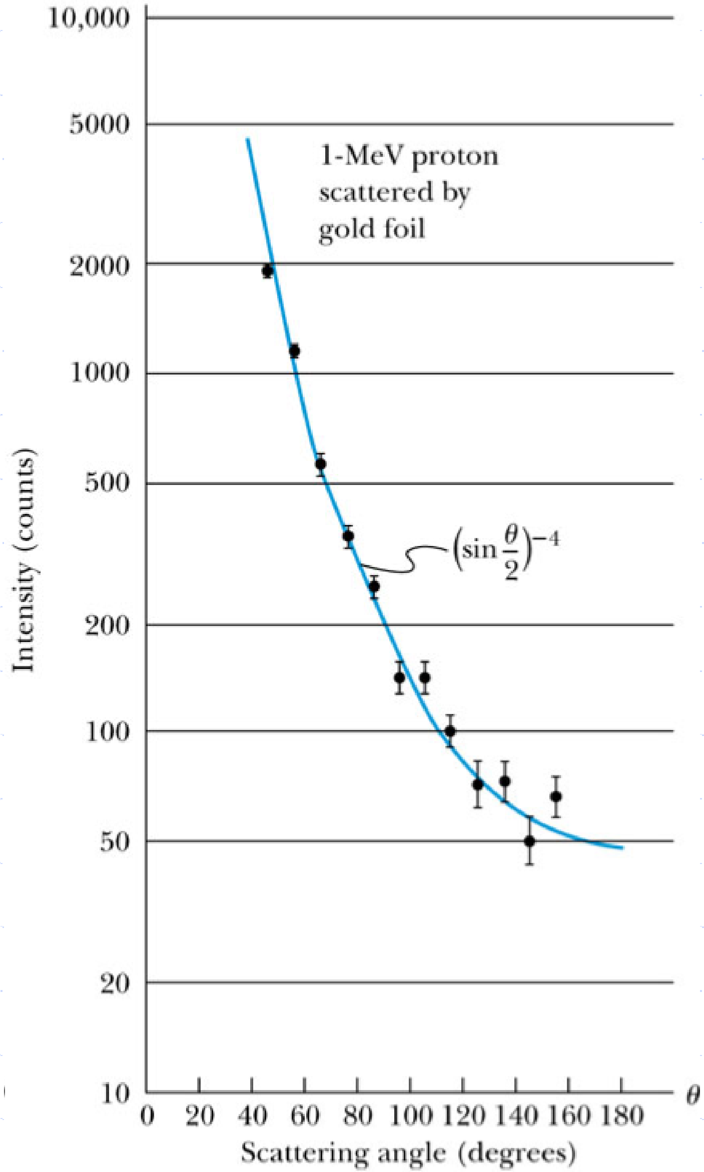
- Sphere diameter ~ 10^{-10}m
- Problems explaining line spectra and electrical stability



Rutherford Scattering



Rutherford Scattering



Rutherford Scattering

- Scattering from electrons?
  - Recall from mechanics that a head-on collision from an alpha with momentum MV with an electron of mass m at rest gives a momentum change of 2mV for the electron
  - The momentum change of the alpha is approximately 2mV=2MV/8000=MV/4000
  - An upper limit for the deflection can be found by assuming the momentum change is perpendicular to the momentum

$$\frac{\Delta p}{p} \approx \tan\theta \approx \theta \approx \frac{1}{4000} \approx 0.01^\circ$$

  - Even multiple scattering from many (thousands) of electrons does not lead to scattering angles of much more an ~1°

Rutherford Scattering

- Scattering from the plum pudding?
  - The maximum Coulomb force on an alpha of charge q a distance R from a charge Q is

$$F = \frac{kqQ}{R^2}$$

  - The momentum change is

$$\Delta p \approx F\Delta t = \frac{kqQ}{R^2}\frac{2R}{V}$$

  - Again taking the change in momentum to be perpendicular to the momentum

$$\tan\theta \approx \theta \approx \frac{\Delta p}{p} = \frac{kqQ}{R\frac{1}{2}MV^2}$$

$$\theta \approx \frac{(2)(79)(14.4eV - A)}{(1A)(5\times 10^6 eV)} = 0.026°$$

Rutherford Scattering

- A convenient form for ke^2

$$ke^2 = \left(9\times10^9 N\frac{m^2}{C^2}\right)\left(1.6\times10^{-19}C\right)^2$$

$$= \left(9\times10^9\right)\left(1.6\times10^{-19}\right)^2 J-m\times\frac{1eV}{1.6\times10^{-19}J}$$

$$= 14.4\times10^{-10} eV-m$$

$$= 14.4eV-Angstroms(A)$$

Rutherford Scattering

- The complete calculation of the Rutherford scattering cross section can be found in Thornton and Rex (section 4.2)
- Instead I will do a poor man's calculation to show the idea
  - Assume a charged particle Z_1e Coulomb scatters from a nucleus of charge Z_2e
  - Assume the nucleus is infinitely massive

Rutherford Scattering

Electric force $F = k\frac{Z_1 Z_2 e^2}{b^2}; k = \frac{1}{4\pi\varepsilon_0}$

Interaction time $\Delta t = \frac{2b}{v}$

Impulse $\mathrm{I} = \Delta p_T \approx F\Delta t = k\frac{2Z_1 Z_2 e^2}{bv}$

Scattering angle $\theta \approx \frac{\Delta p_T}{p} = k\frac{2Z_1 Z_2 e^2}{pbv}$

So $b = k\frac{2Z_1 Z_2 e^2}{pv\theta}$

Aside, the exact relation is $b = k\frac{Z_1 Z_2 e^2}{pv}\cot\frac{\theta}{2}$

And note $\cot\frac{\theta}{2} \approx \left(\frac{\theta}{2}\right)^{-1} - \frac{1}{3}\frac{\theta}{2} - \ldots$

Rutherford Scattering

Then $\frac{db}{d\theta} = -k\frac{2Z_1Z_2e^2}{pv\theta^2}$

Recalling our earlier result $\frac{d\sigma}{d\Omega} = \left|\frac{b}{\sin\theta}\frac{db}{d\theta}\right|$

And using $\sin\theta \approx \theta$ and $b = k\frac{2Z_1Z_2e^2}{pv\theta}$

We arrive at

$$\frac{d\sigma}{d\Omega} = k^2\frac{4Z_1^2Z_2^2e^4}{p^2v^2}\frac{1}{\theta^4}$$

That you can compare to the exact Rutherford scattering cross section

$$\frac{d\sigma}{d\Omega} = k^2\frac{Z_1^2Z_2^2e^4}{p^2v^2}\frac{1}{4\sin^4\left(\frac{\theta}{2}\right)}$$

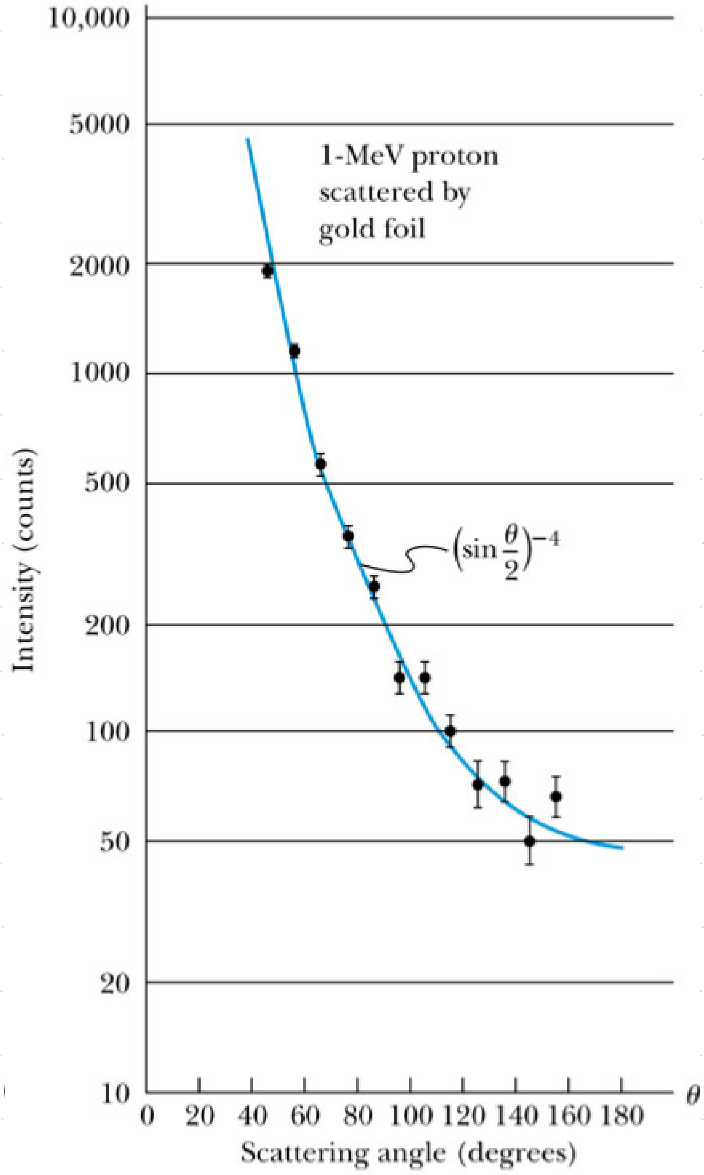
Rutherford Scattering

- Rutherford scattering differential cross section

$$\frac{d\sigma}{d\Omega} = k^2 \frac{Z_1^2 Z_2^2 e^4}{p^2 v^2} \frac{1}{4\sin^4\left(\frac{\theta}{2}\right)}$$

- The important features are
  - $1/\sin^4(\theta/2)$ dependence
  - Z_1^2, Z_2^2 dependence
  - $1/T^2$ dependence
- All observed experimentally

Rutherford Scattering



Rutherford Scattering

- Consider scattering from the plum pudding
  - We estimated the scattering angle to be θ~0.026°
  - Invoking the central limit theorem-the sum of a large number of independent variables approaches a Gaussian distribution

$$N = N_0 e^{-\left(\frac{\theta}{\theta_m}\right)^2}$$

- The expected number scattered through 90° or more is

$$N_{90} = N_0 e^{-\left(\frac{90}{1}\right)^2} = N_0 e^{-8100} \approx N_0 10^{-3500} \approx 0$$

Rutherford Scattering

- The relation between impact and scattering angle is

$$b = \frac{Z_1 Z_2 e^2}{8\pi\varepsilon_0 T}\cot\frac{\theta}{2}$$

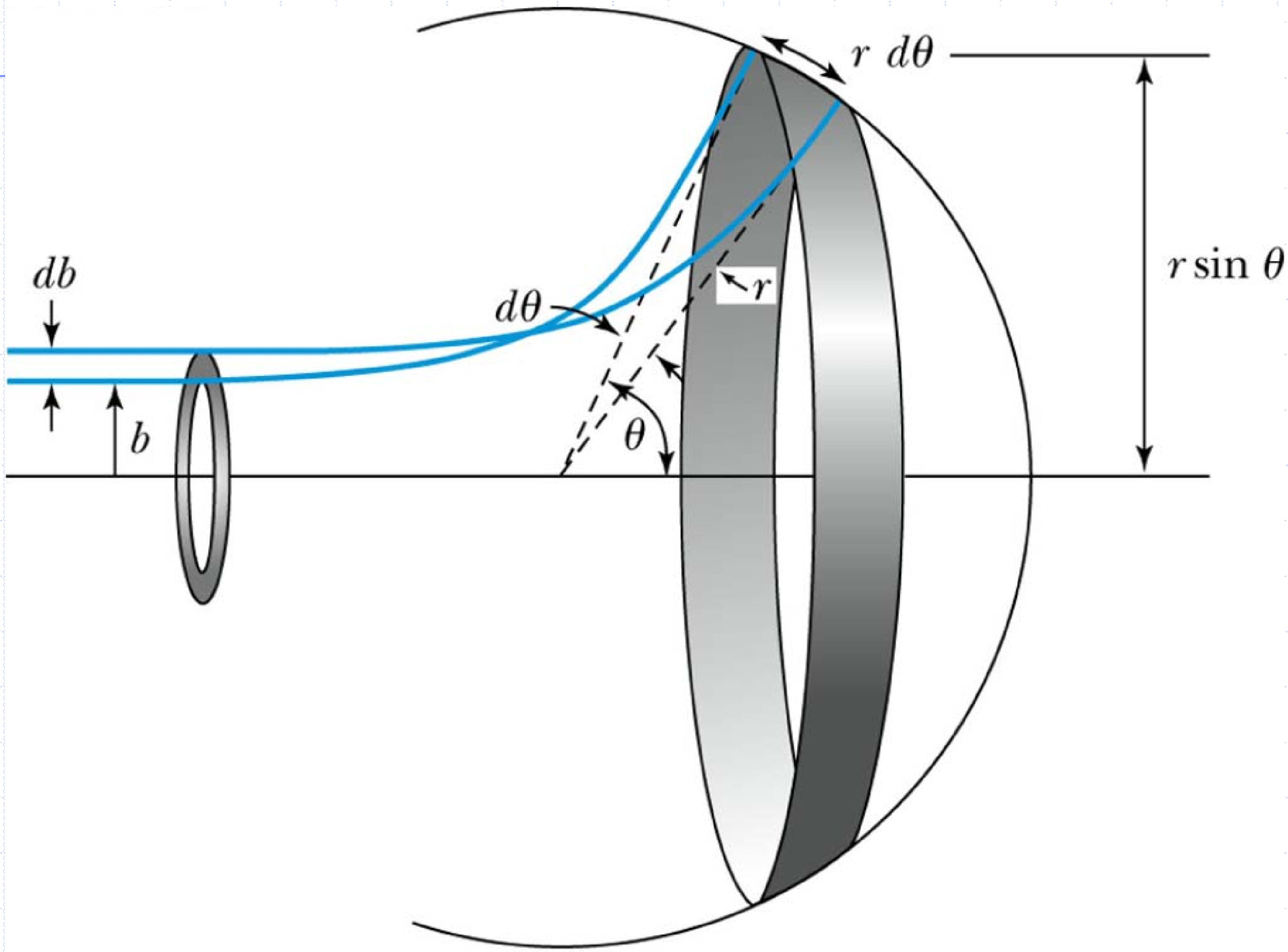
- The scattering is deterministic
  - This means all alpha particles with impact parameter < b will scatter into angles > θ
  - So the number of alpha particles scattered into angles > θ is

$$\pi b^2 \times N_T$$

where N_T is the number of target nuclei/area

$$N_T = N_{Av}\rho t / A$$

Cross Section



Rutherford Scattering

- What fraction of particles is scattered through angles > 90°
  - 7.7 MeV alpha incident on an Au target of thickness $t=10^{-6}$m

$$N_T = \frac{\left(6.02\times10^{23}\right)\left(19.3\frac{g}{cm^3}\right)\left(10^{-6}m\right)}{197\frac{g}{mol}} = 5.9\times10^{22}\frac{nuclei}{m^2}$$

$$f = \pi b^2 \times N_T$$

$$f = \pi\left(5.9\times10^{22}\right)\left(\frac{(79)(2)(14.4eVA)}{(2)(7.7MeV)}\right)^2 \cot^2 45^\circ$$

$$f = 4\times10^{-5}$$

Rutherford Scattering

- How large is the nucleus?
- Consider a head-on collision (180° scattering) of a 5 MeV alpha with a gold nucleus
- The distance of closest approach is found from

$$\frac{kqQ}{d} = \frac{1}{2}mV^2$$

$$d = \frac{kqQ}{\frac{1}{2}mV^2}$$

$$d = \frac{(2)(79)14.4eV - A}{5MeV} = 4.5 \times 10^{-14}m = 45F$$

Rutherford Scattering

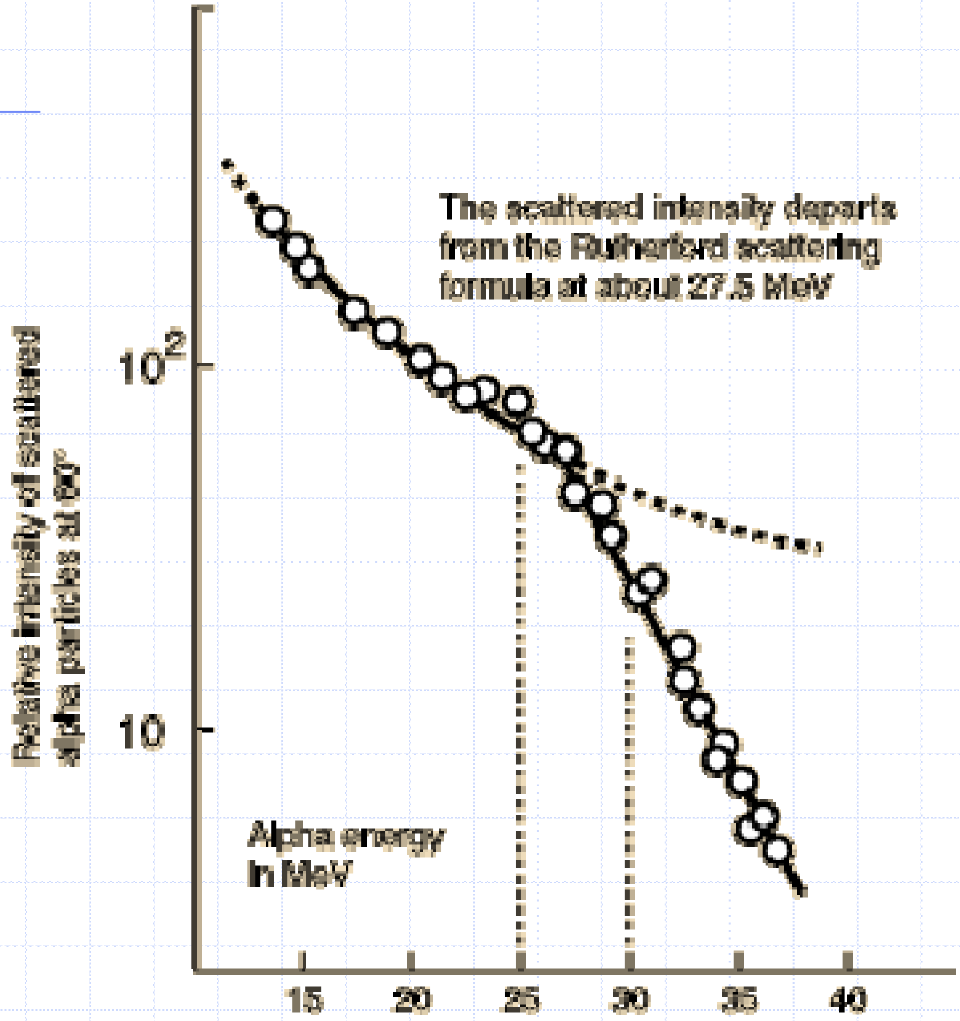
- The total cross section for Rutherford scattering is infinite!

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

$$\sigma = k^2 \frac{Z_1^2 Z_2^2 e^4}{4 p^2 v^2} \int_0^{\pi} \frac{1}{\sin^4\left(\frac{\theta}{2}\right)} \sin\theta d\theta d\varphi$$

$$\sigma = \infty$$

Rutherford Scattering



- Welcome to nuclear physics