Relativistic Momentum

- Question: Would you expect Newton's second law F=ma to hold at high velocity?
- Answer: No, a constant force can accelerate a particle to $v>c$
- Answer: No, certainly something is different since in a Galilean transformation the force and acceleration are the same. This is not true for a Lorentz transformation.

Newtonian Relativity

- Note Newton's laws are valid in both frames
  - The force and acceleration are the same in both frames
  - There is no way to detect which frame is moving and which is at rest

$$F'_y = ma'_y = m\frac{d^2y'}{dt'^2} = m\frac{d^2y}{dt^2} = ma_y = F_y$$

$$F'_x = ma'_x = m\frac{d^2x'}{dt'^2} = m\frac{d^2(x-vt)}{dt^2} = m\frac{d^2x}{dt^2} = F_x$$

Relativistic Momentum

- Galilean transformation
  - We showed *a=a′* and *F=F′*
- Lorentz transformation

$$u_x = \frac{u'_x + V}{1 + \frac{V u'_x}{c^2}}$$

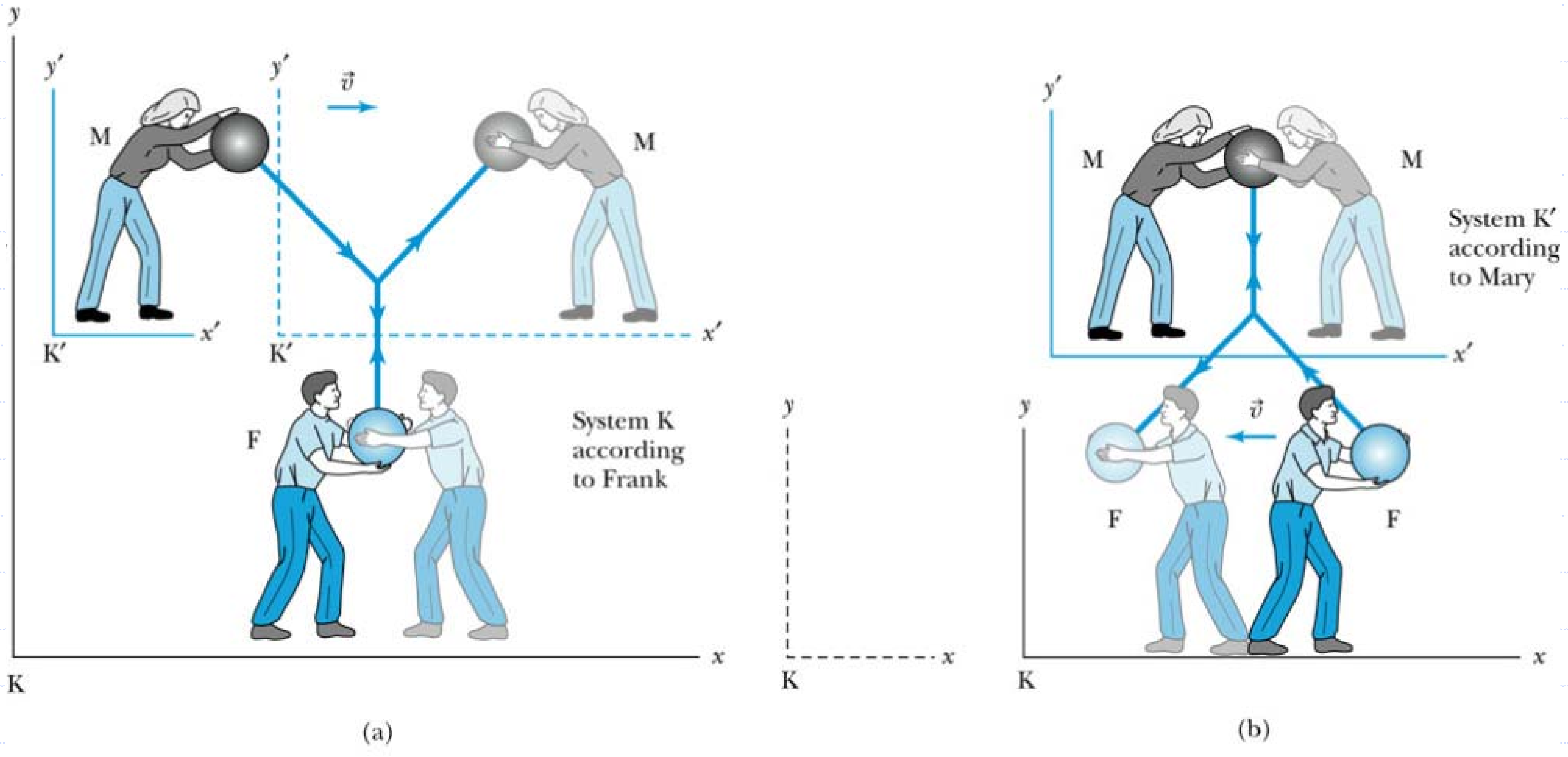
$$a_x = \frac{du_x}{dt} = \frac{a'_x}{\gamma^3 \left(1 + \frac{V u'_x}{c^2}\right)^3}$$

- We might try starting with *F=dp/dt*

Relativistic Momentum

- Frank (F) is in K with a ball of mass m
- Mary (M) is in K' with a ball of mass m
- Frank throws his ball along y with velocity u_0
- Mary throws her ball along –y' with velocity u_0
- The balls collide elastically

Relativistic Momentum



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Relativistic Momentum

- Frank sees the momentum change of his ball

$$\Delta p_{Fx} = 0$$

$$\Delta p_{Fy} = -2mu_0$$

- Frank sees for Mary's ball

$$u'_x = 0 \text{ and } u'_y = -u_0$$

$$\text{thus } u_x = V \text{ and } u_y = -u_0\sqrt{1-\frac{V^2}{c^2}}$$

- Frank sees the momentum change of Mary's ball

$$\Delta p_{Mx} = mV - mV = 0$$

$$\Delta p_{My} = mu_0\sqrt{1-\frac{V^2}{c^2}} + mu_0\sqrt{1-\frac{V^2}{c^2}}$$

$$= 2mu_0\sqrt{1-\frac{V^2}{c^2}}$$

Addition of Velocities

- Recall the addition of velocities in special relativity

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + v dt')}{\gamma\left[dt' + \left(\frac{v}{c^2}\right)dx'\right]} = \frac{u'_x + v}{1 + \left(\frac{v}{c^2}\right)u'_x}$$

similarly

$$u_y = \frac{u'_y}{\gamma\left[1 + \left(\frac{v}{c^2}\right)u'_x\right]}$$

$$u_z = \frac{u'_z}{\gamma\left[1 + \left(\frac{v}{c^2}\right)u'_x\right]}$$

Relativistic Momentum

- Momentum is not conserved in the y direction!
- Because we strongly believe in the conservation of momentum, let's modify the definition of momentum

$$\vec{p} = \gamma m \vec{u}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Relativistic Momentum

- Frank sees the momentum change of his ball

$$\Delta p_{Fy} = \frac{-2mu_0}{\sqrt{1-\frac{u_0^2}{c^2}}}$$

- Frank sees the momentum change of Mary's ball

$$\Delta p_{My} = 2\gamma m u_0 \sqrt{1-\frac{V^2}{c^2}}$$

$$\text{where } \gamma = \frac{1}{\sqrt{1-\frac{\mathrm{V}^2 + u_0^2\left(1-\frac{V^2}{c^2}\right)}{\mathrm{c}^2}}} = \frac{1}{\sqrt{\left(1-\frac{u_0^2}{c^2}\right)\left(1-\frac{V^2}{c^2}\right)}}$$

$$\Delta p_{My} = \frac{2mu_0}{\sqrt{1-\frac{u_0^2}{c^2}}}$$

Relativistic Momentum

- Thus Frank sees that momentum is conserved in the x and y directions using

$$\vec{p} = \gamma m \vec{u}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

- Likewise Mary would see that momentum conservation holds in her frame as well

Relativistic Momentum

- Notes
  - Unfortunately γ is used for both

$$\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \text{ and } \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

  - Usually we write them out when they both come into play
  - In these equations $m=m_0$ is the rest mass
  - Sometime we interpret γm as the relativistic mass but that is not standard

Relativistic Energy

- Let’s assume that the relativistically correct form of Newton’s law is given by

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d\left(\gamma m\vec{u}\right)}{dt}$$

- The validity of this assumption can be determined by examining its consequences
- Aside, are Newton’s first and third laws relativistically correct?

Relativistic Energy

- In classical mechanics the work done by a force in moving a particle from one position to another equals the change in kinetic energy

$$T = \int_{u=0}^{u} F ds = \int_{0}^{u} \frac{d(\gamma m u)}{dt} ds = \int_{0}^{u} u d(\gamma m u)$$

$$d(\gamma m u) = m\gamma du + m u d\gamma$$

Relativistic Energy

- We can evaluate the integral by writing

$$d\gamma = d\left(1-\frac{u^2}{c^2}\right)^{-1/2} = \frac{u}{c^2}\left(1-\frac{u^2}{c^2}\right)^{-3/2} du$$

$$d(\gamma m u) = m\gamma du + mu d\gamma$$

$$= m\left[\left(1-\frac{u^2}{c^2}\right)^{-1/2} du + \frac{u^2}{c^2}\left(1-\frac{u^2}{c^2}\right)^{-3/2} du\right]$$

$$= m\left(1-\frac{u^2}{c^2}\right)^{-3/2} du$$

Relativistic Energy

- Then

$$T = \int_0^u u\,d(\gamma m u) = \int_0^u m\left(1-\frac{u^2}{c^2}\right)^{-3/2} u\,du$$

$$T = mc^2\left[\left(1-\frac{u^2}{c^2}\right)^{-1/2} - 1\right]$$

- Thus $T = \gamma mc^2 - mc^2$

Relativistic Energy

- Now at low speeds we can use the binomial expansion

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots$$

- To find as expected

$$T = mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots - 1\right) = \frac{1}{2}mu^2$$

Relativistic Energy

- We define the total energy E
  - E = kinetic energy plus rest energy
  - $E = T + mc^2$
  - $E = \gamma mc^2$

Relativistic Energy

- We also have for *E*

$$E^2 = \gamma^2 m^2 c^4$$

$$p^2 c^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 \frac{u^2}{c^2} c^4$$

$$E^2 - p^2 c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{u^2}{c^2} \right) = m^2 c^4$$

- Thus
  - $E^2 = p^2 c^2 + m^2 c^4$
  - $E = mc^2$ for $p = 0$