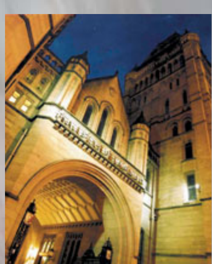
Linear Accelerators: Theory and Practical Applications: WEEK 5

Stanford Linear Accelerator, shown in an aerial digital image. The two roads seen near the accelerator are California Interstate 280 (to the East) and Sand Hill Road (along the Northwest). Image data acquired 2004-02-27 by the United States Geological Survey



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Summary of Week 4

- Coupled cavity linacs operating in the pi/2 mode were described with details provided on means to achieve high-efficiency of acceleration – comparing favourably with pi mode acceleration.
- General phase stability criterion and 'fish' diagrams were discussed.
- Transient beam loading was derived for single and multi-bunch beams.
- Approximate 'zero order' design equations were developed leading to an analytical formula relating the iris radius a, and cavity radius b, in a disk loaded slow wave structure to the synchronous particle beam velocity ($v_p \sim c$), at a prescribed frequency ω.
- Experimental means to determine loss factor (= $\omega R/4Q$) were described via a wire measurement –a bench top measurement that is relatively inexpensive to perform (c.f. measurements performed in a high energy linac facility such as SLAC)
- General expressions for multi-cell loss factors were delineated in terms of single cell loss factors –important for practical measurements and simulations.

Overview of Week 5

- **Multi-bunch beam loading analysis –early injection means to compensate for field droop due to beam loading.**
- **Practical multi-bunch parameters.**
- **Brief introduction to longitudinal and transverse wake-fields and their influence on beam dynamics.**
- **The practical significance of short-range and long-range wake-fields is discussed.**
- **Means to compensate for the energy spread induced across the beam –longitudinal wake-field compensation**
- **Influence of transverse wake-field on charged bunch transport – BBU (Beam Break Up) and emittance dilution.**
- **Detailed analysis of two macro particle model of BBU**
- **Means to ameliorate BBU via BNS damping**
- **Practical means to cope with both longitudinal wake-field compensation and BNS damping of transverse wake-field-induced instabilities.**

Early Injection for Multi-bunch Operation

- In order to increase the luminosity and RF energy transfer efficiency of a linear collider, multi-bunch operation will almost certainly be required.
- The beam should then be injected before the accelerator section is completely filled so that to first order, the energy decrease due to beam loading is compensated by the energy increase due to filling.
- <u>Homework.</u> For a bunch train consisting of bunches spaced by ΔS from their neighbours show that the energy compensation condition and maximum energy sag are given by:

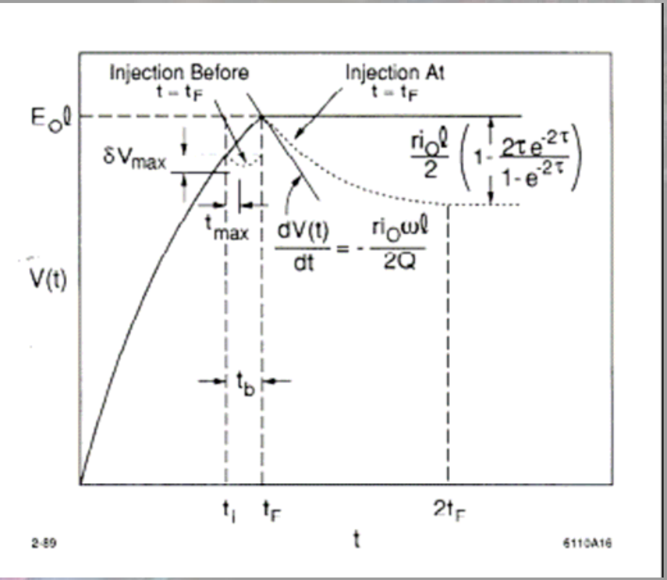
$$\frac{\Delta S}{L} = \left(\frac{ct_F}{L}\right)\frac{2qk_{loss}/L}{E_0 + Nqk_{loss}/L}$$

$$\delta V_{max} = -\frac{\tau}{2(1-e^{-2\tau})}\left(\frac{1}{ct_F}\right)N^2 k_{loss} q \Delta S$$

Early Injection for Multi-bunch Operation

➢We assume in this example that the beam injection starts at time $t_i = t_F - t_b$ and ends at time t_F.

➢To obtain the best energy compensation we choose to assume that the following condition must be satisfied: $V(t_i) = V(t_F)$



The general expression we derived for the energy gain:

$$V(t) = \frac{E_0 L\left[1 - e^{-(\omega/Q)t}\right]}{\left(1 - e^{-2\tau}\right)} U(t) - \frac{E_0 L e^{-2\tau}}{\left(1 - e^{-2\tau}\right)}$$

$$\left[1 - e^{-(\omega/Q)(t - t_f)}\right] U(t - t_f) + \frac{r i_0 L}{2}\left\{\frac{\omega L e^{-2\tau}}{Q(1 - e^{-2\tau})}(t - t_i) - \frac{L}{(1 - e^{-2\tau})}\left[1 - e^{-(\omega/Q)(t - t_i)}\right]\right\} U(t - t_i)$$

under the condition $V(t_i) = V(t_F)$, yields:

$$\frac{E_0\left[1 - e^{-(\omega/Q)t}\right]}{\left(1 - e^{-2\tau}\right)} = E_0 + \frac{r i_0}{2}\left[\frac{2\tau e^{-2\tau}}{1 - e^{-2\tau}} \frac{t_b}{t_F} - \frac{1 - e^{-2\tau t_b / t_F}}{1 - e^{-2\tau}}\right]$$

Early Injection for Multi-bunch Operation

Assuming $t_b << t_F$ and expanding exponentials yields:

$$\text{energy compensation conditions } E_0 + \frac{ri_0}{2}\left[\left(1-e^{2\tau}\right)+\tau\left(\frac{t_b}{t_F}\right)e^{2\tau}\right]=0$$

The energy deviation along the bunch train at time t^* is given by:

$$V(t_i+t^*)=\frac{E_0L}{\left(1-e^{-2\tau}\right)}\left[1-e^{-(\omega/Q)(t_F-t_b+t^*)}\right]+\frac{ri_0L}{2(1-e^{-2\tau})}\left[\frac{\omega e^{-2\tau}}{Q}t^*-\left[1-e^{-(\omega/Q)t^*}\right]\right]$$

where the general expression for energy has been used.

Expanding the exponentials:

$$V(t_i+t^*)=\frac{E_0L}{\left(1-e^{-2\tau}\right)}\left[1-e^{-2\tau}\left[1+\frac{2\tau(t_b-t^*)}{t_F}\right]\right]+\frac{ri_0L}{2(1-e^{-2\tau})}\left[\frac{\omega e^{-2\tau}}{Q}t^*-2\tau\left(\frac{t^*}{t_F}\right)+2\tau^2\left(\frac{t^*}{t_F}\right)^2\right]$$

V_0, the energy of the first bunch is obtained by setting $t^*=0$:

$$V_0=\frac{E_0L}{\left(1-e^{-2\tau}\right)}\left[1-e^{-2\tau}\left[1+2\tau\left(\frac{t_b}{t_F}\right)\right]\right]$$

Early Injection for Multi-bunch Operation

$$\delta V = V(t_i + t^*) - V_0$$

$$= \frac{2E_0 L\tau e^{-2\tau}}{\left(1-e^{-2\tau}\right)}\left(\frac{t^*}{t_F}\right) + ri_0 L\tau\left[-\frac{t^*}{t_F} + \frac{\tau}{1-e^{-2\tau}}\left(\frac{t^*}{t_F}\right)^2\right]$$

The time at which the minimum energy deviation occurs is obtained from:

$$\frac{d\delta V}{dt^*} = 0$$

$$\Rightarrow t^* = \left(\frac{1-e^{-2\tau}}{2\tau} - \frac{e^{-2\tau}E_0}{\tau r i_0}\right)t_F$$

Using energy compensation condition reveals the maximum energy deviation

occurs at $t_{max} = \frac{t_b}{2}$

Thus, the maximum sag is located in the centre of the bunch train. Substituting this value for t_{max} in δV and using the energy compensation condition gives:

$$\delta V_{max} = -ri_0 L \frac{\tau^2}{4\left(1-e^{-2\tau}\right)}\left(\frac{t_b}{t_F}\right)^2$$

Energy Deviation in terms of Multi-bunch Parameters

The charged particles in an RF linac are rigidly and regularly bunched.

We assume N bunches, equally spaced from their neighbours by ΔS. Under these conditions we have the following basic relations:

$i_0 t_b = Nq$

$N\Delta S = ct_b$

$$\tau = \frac{\omega}{2Q} t_F = 2\left(\frac{\omega r}{4Q}\right)\frac{t_F}{r} = \frac{2k_{loss} t_F}{r}$$

These relations enable the energy compensation condition to be written as:

$$E_0 + Nk_{loss} q e^{2\tau} = k_{loss} q \left(\frac{Nt_F}{t_b}\right)\left(\frac{e^{2\tau}-1}{\tau}\right)$$

Also, using $t_b = \frac{\Delta S}{L}\left(\frac{NL}{c}\right)$ gives the energy compensation as:

$$\frac{\Delta S}{L} = \left(\frac{e^{2\tau}-1}{\tau}\right)\left(\frac{ct_F}{L}\right)\frac{k_{loss} q}{E_0 e^{-2\tau} + Nk_{loss} q}$$

Energy Deviation in terms of Multi-bunch Parameters

The nth bunch has maximum energy sag:

$$t_{max} = \frac{t_b}{2} \text{ and thus } n \approx \frac{N}{2}$$

The energy deviation of this bunch can be expressed as:

$$\delta V_{max} = -\frac{\tau}{2\left(1-e^{-2\tau}\right)}\left(\frac{L}{ct_F}\right)N^2 k_{loss} q \Delta S$$

Also, for small attentuation the energy compensation condition becomes:

$$\frac{\Delta S}{L} = \left(\frac{ct_F}{L}\right)\frac{2k_{loss}q}{E_0 + Nk_{loss}q}$$

Similarly making a taylor expansion in τ the maximum energy deviation is:

$$\delta V_{max} = -\left(\frac{L}{ct_F}\right)\frac{N^2}{4}k_{loss}q\Delta S$$

Wakefields, Energy Loss and BBU

- The fields scattered by obstacles in the accelerator constitute the wake-field.
- Fourier transform of the wake-field is the beam impedance.
- The wake-field results from, in principle an infinite series of modes excited in the cavity or waveguide.
- These modes are decomposed into a discrete sum (and supplemented with an analytic extension as appropriate)
- The wake-field has both longitudinal and transverse vector components.
- The longitudinal wake-field affects the energy along the bunch –it leads to an energy spread
- The transverse wake-field kicks the bunch (or series of bunches) perpendicular to the direction of transport.
- The transverse wake-field can give rise to a BBU (Beam Break Up) instability which disrupts the progress or a minimum can dilute the emittance and hence the luminosity of the beam.

Wakefields, Energy Loss and BBU

- **Wake-field is usually decomposed into its intra-bunch, or short range and, long-range components.**
- **Long-range wake-field is usually well-represented with a limited number of modes.**
- **Short-range wake-field required several hundred modes and is often calculated in the time domain**

Single Bunch Beam Loading

How does a point charge induce field and lose energy to a cavity?

Assume a point charge q moves along the z-axis the field is related to the energy according to:

$$E_z(z) = \alpha(z)\sqrt{U}$$

A change in mode energy dU changes the field E_z :

$$2E_z dE_z = \alpha^2 dU$$

$$\Rightarrow dE_z(z) = \frac{\alpha^2(z)}{2E_z} dU$$

Also, charge loses energy:

$$dU_q = -qE_z dz$$

By conservation of energy the $dU = dU_q$

$$\Rightarrow dE_z(z) = -\frac{1}{2} q\alpha^2(z) dz$$

Single Bunch Beam Loading

The field increases as the beam transits the cavity.

Define a reference position at z=0, $\alpha(0)=\alpha_0$, $E_z(0)=E_0$ and assume z=ct:

$$dE_0(t)=\left[\alpha_0/\alpha(z)\right]dE_z(z)=-\frac{1}{2}qc\left[\alpha_0\alpha(ct)\right]dt$$

A field element induced at time t' will give rise to:

$$d\tilde{E}_0(t)=dE_0(t')e^{j\omega(t-t')}$$

By superposition the field induced as the charge transits from 0 to L through the cavity:

$$\tilde{E}_{0b}(t=L/c)=-\frac{1}{2}q\alpha_0\int_0^L\alpha(z')e^{jk_0(L-z')}dz';\ \ k_0=\omega/c$$

$$\tilde{E}_{0b}=-\frac{1}{2}q\frac{\alpha_0 e^{jk_0L}}{\sqrt{U}}\int_0^L E_z(z')e^{jk_0(L-z')}dz'$$

The voltage wrt the beam is given by:

$$\tilde{V}=\int_0^L E_z(z')e^{jk_0z'}dz'$$

Single Bunch Beam Loading

$$\tilde{E}^*_{0b} = -\frac{q\alpha_0}{2\sqrt{U}} e^{-jk_0L} \int_0^L E_z(z')e^{jk_0z'}dz'$$

$$\tilde{E}^*_{0b}\tilde{E}_{0b} = q^2\alpha_0^2 . \frac{\tilde{V}_b\tilde{V}_b^*}{4U}$$

But $\dfrac{E_0^2}{\alpha_0^2} = U$

Define $k_{loss} = \dfrac{\tilde{V}_b\tilde{V}_b^*}{4U}$

$\Rightarrow U = k_{loss}q^2$

Thus, the loss factor describes the total energy lost by the chargeper unit charge.

Similarly, one can show that:

$V_b = 2k_{loss}q$

The total voltage induced by the beam

Wakefields, Energy Loss and BBU

- The longitudinal wake of a particular mode is calculated according to: $W_z(s)=2k_{loss,0}\cos(\omega_o s/c)$. The complete wake-field is obtained by summing over all such modes.

- For the transverse modes, the dipole is the most critical. The dipole wake is computed according to:

$$W_t(s) = \frac{2k_{loss,d} r / a_0}{(\omega a_0 / c)} \sin(\omega_d s / c)$$

where r is the offset of the driving bunch, a_0 is the position at which the loss factor is calculated, $\omega_d/(2\pi)$ is dipole mode frequency and $k_{loss,d}$ is the dipole mode loss factor. Again, we sum over all modes to obtain the complete field.

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1. Short-Range Wake-fields

Short-Range Wake-fields and Parasitic Losses

- HOMs do work on the beam, causing a parasitic energy loss. The lost beam energy can be made up for by increasing the applied RF voltage.

- Wakefield energy is also lost in the walls of the cavity via Ohmic eating. It is important to consider if the walls can be cooled sufficiently to cope with this increase in temperature. We will not deal with the heat load specifics. Cu structures are wate cooled. Maintaining the temperature of superconducting structures (Niobium) is a more serious issue in general.

- The energy lost to the wake-fields per cell is computed using the loss-factor:
  $\Delta U(L)=(eN)^2 k_{tot}$ (N= # of electrons in a bunch)
  and for a linac of section length L with a cell of length l_c:
  $\Delta U(L)=(eN)^2 k_{tot} L/l_c$ ----Energy lost per section.

- For a given bunch repetition rate f_R the average power lost to the wakefields is:
  $P_{av}(L)= \Delta U(L) f_R= f_R (eN)^2 k_{tot} L/l_c$ ------- Power lost to wake-fields
- For the SLAC 2 mile linac: $N \sim 5 \times 10^{10}$, k_{tot} ~' 2×10^{12} V/C, L= 1m, l_c = 3.5 cm (= $\lambda/3$), f_R = 120 Hz => P_{av} = 0.44 W/m.

Short-Range Wake-fields and Parasitic Losses

- A power loss of 0.44 W/m is a small heat load for cu structures but it is a substantial, although manageable load for Ni superconducting accelerating cavities. Finally, over the complete linac, P_{tot} ~ 3000 x 0.44 = 1.32 kW.

- How can we reduce these parasitic losses? Clearly, reducing the total loss factor, facilitated by increasing the iris aperture will achieve this. However, the fundamental mode loss factor will also be reduced concomitantly. This increases the voltage requirements as the power coupled to the beam is given by: I^2R ($k_{loss} = \omega R/(4Q)$). Often we choose to go in the other direction, reduce iris size, and hence increase the overall loss factor in order to efficiently couple to the accelerating mode –the HOMs are then carefully damped.

- It is worth bearing in mind that various other components, other than cavities, which often contribute significantly to the overall impedance and must not be ignored –for example vacuum belows, diagnotic BPMs, fundamental mode couplers and tuners.

- Other than power dissipation, why are these longitudinal HOMs a cause for concern? If they are not damped, then they can give rise to an energy spread along each bunch and along the bunch train.

Short-Range Wake-fields and Energy Spread

- For linear colliders and FELs energy spread must be minimised.

- It is often a good approximation to assume a linear dependence of the short range wake along the bunch. For a cell length l_c and a linac section of length L the energy loss is given by:

$$\Delta W(s,L) = -\frac{eQW_z^{'}sL}{l_c}$$

- Particles at the head of the bunch suffer minimal energy loss. However, the head drives the tail and hence a significant energy loss occurs in the tail. The energy spread across the bunch is given by:

$$\delta W(L) = \frac{eQW_z^{'}s_{tail}L}{l_c}$$

- If nothing is done to correct for this energy spread then considerable chromatic aberations can occur in the final output beam

Short-Range Wake-fields and Beam Loading Compensation

- Fortunately there is a relatively straightforward means of reducing the energy spread along the bunch. If we accelerate below the crest of the RF field then slower particles will arrive later and will gain more energy than the head. Thus, we choose the phase of the centroid of the bunch accordingly.

- We arrive at a condition for this phase by considering the energy loss due to beam loading and the energy gain due to operating behind the crest:

$\Delta\omega(\Delta\phi, L) = -eQ(W_z' c/\omega)\,\Delta\phi\, L/l_c$ (where $\Delta\phi = \omega s/c$)
$\Delta\omega_{RF} = \Delta(eEL\cos\phi) \sim -eEL\sin\phi_0\,\Delta\phi$ (at some particular phase $\phi = \phi_0$)

- $$\Rightarrow \sin\phi_0 = -\frac{QW_z' c/\omega}{El_c}$$

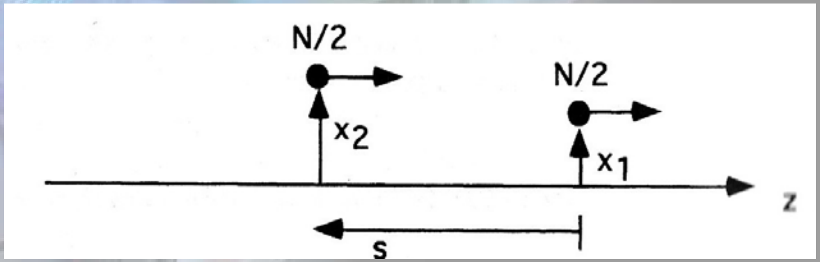
- This defines the beam centroid phase. Now, as Wz' is the derivative of the wake-field then larger longitudinal wake gradients will require larger Abs(ϕ) to properly compensate for beam loading. This impacts the overall efficiency of acceleration.

Single Bunch Beam Break Up (BBU) due to Short Range Wake-fields

- **BBU occurs as a result of transverse deflecting modes. For small offsets, the dipole mode dominates. It was first observed in the late 1950s and early 1960s on the SLAC linac in California.**

- **The instability is well described by model which utilises two macro-particles, representing one bunch, containing N particles. Each macro-particle contains N/2 particles, separated from each other by a distance s.**

- **The leading macro-particle is offset and executes betatron motion about the design orbit in the focussing quadrupole lattice.**



Parameters of Two-Particle Model of BBU

- The full motion of a bunch of particles in a focussing lattice includes rapid ripples. Here, we use a smooth approximation in the particles execute SHM. The driving particle is described by:
  $x_1 = X \cos \omega_\beta t$.
  The wakefield left behind the driving particle modifies the motion of the trailing particle. Over a cell of length l_c the average force F_w produced by the head of charge Ne/2 on an electron e in the tail evaluates to:

$$F_W l_c = \Delta p_\perp c = e^2 \frac{N}{2} W_\perp(s)$$

It is convenient to write the wake-field force as:

$$F_W = \frac{e^2 N}{2 l_c}\left[\frac{W_\perp}{x_1 / a}\right]\frac{x_1}{a}$$

The equation of motion for particles in the tail in the approximation of a constant energy beam is:

$$m\gamma \ddot{x}_1 = -m\gamma\omega_\beta^2 x_1$$

$$m\gamma \ddot{x}_2 = -m\gamma\omega_\beta^2 x_2 + F_W$$

or $$\boxed{\ddot{x}_2 + \omega_\beta^2 x_2 = \frac{e^2 N}{2 m\gamma l_c a}\left[\frac{W_\perp}{x_1 / a}\right] X \cos \omega_\beta t}$$

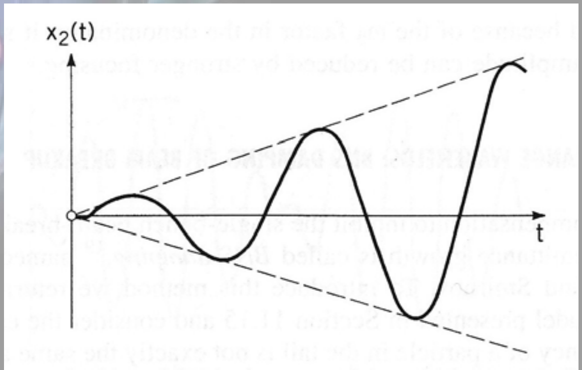
- In this approximation the energy is set equal to a constant value (i.e. γ = constant). The general solution to this SHM driven on resonance is:

$$\frac{x_2}{X} = \cos\omega_\beta t + \frac{e^2 N t}{4 m\gamma l_c a\omega_\beta} \frac{w_\perp}{x_1 / a} \sin\omega_\beta t$$

- The first term is the free (uncoupled) oscillation and the second term is a particular solution of the inhomogeneous equation. The latter term is driven by the wake-field. The amplitude of this term grows linearly with t and at the end of the linac the growth is characterised by:

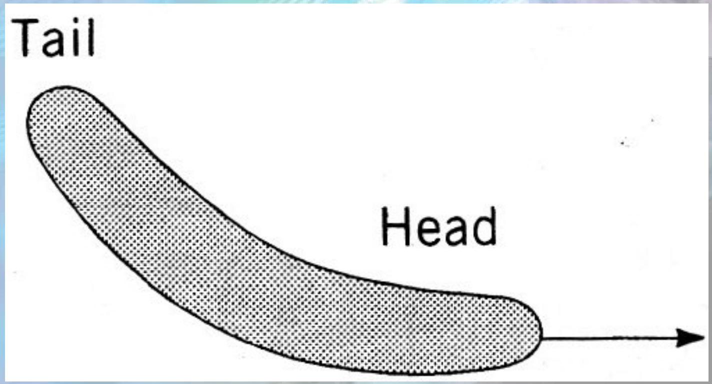
- $$A(s) = \frac{e^2 N L c}{4(\gamma m c^2) l_c a\omega_\beta} \frac{w_\perp}{x_1 / a}$$

The energy was assumed to be constant and in practise there is of course an energy gain down the linac. To account for this energy gain the factor γmc^2 averaged over the length of the linac. For example in the linear approximation:
$\langle\gamma mc^2\rangle = (1/2)mc^2 \langle \gamma_1 + \gamma_2 \rangle$



Resonant growth in the tail of a bunch tail down the linac.

- This effect can lead to a highly unstable beam which breaks up and is lost to the walls of the accelerator. Typical motion of the tail relative to the head is illustrated below. Finally, we note that since the amplitude is inversely proportional to ω_β then increasing the focussing parameter will reduce the amplitude of the oscillation and minimise the likelyhood of BBU. However, there are practical limitations on how large we can make the field in a magnetic –which also consumes power and requires cooling.



Motion of tail of bunch relative to head

Damping BBU Caused by Short Range Wake-Fields

- In linear collider BNS damping is used to minimise the amplitude of the driven oscillations and hence minimise the likelyhood of BBU and associated emittance growth. The technique is named after the authors of the technique –Balakin, Novokhatsky and Smirnov. It relies on the front of the bunch oscillating with a different betaron frequency than the tail. The motion is described by a simple harmonic oscillator driven off-resonance:

$$\ddot{x}_2 + \omega_{\beta 2}^2 x_2 = \frac{e^2 N}{2m\gamma l_c a}\left[\frac{W_\perp}{x_1/a}\right] X\cos\omega_{\beta 1} t$$

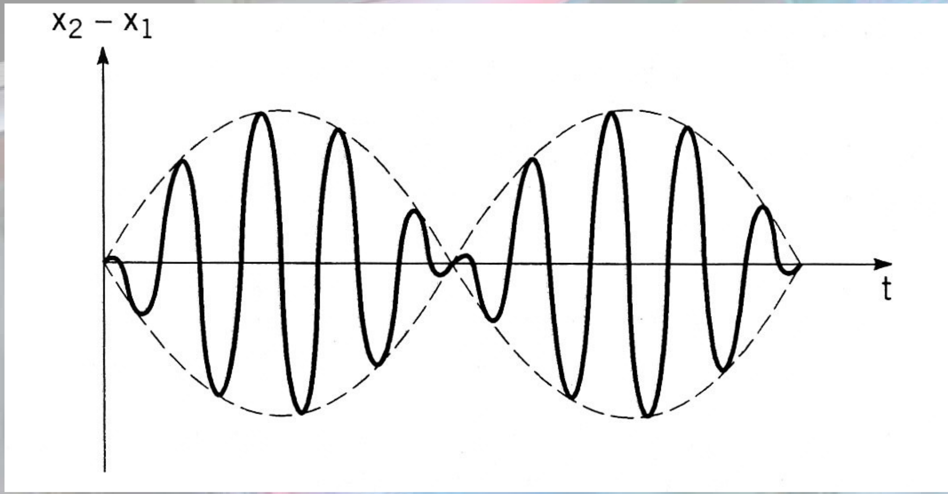
where $\omega_{\beta 1}$ ($\omega_{\beta 2}$) is the betatron frequency of the head (tail). The trajectory of the head of the charge is again given by $x_1 = X\cos\omega_{\beta 1}t$. Taking the initial condition that at $t=0$: $x_1(0) = x_2(0)$ and $dx_1(0)/dt = dx_2(0)/dt$ and solving the driven SHM:

$$x_2 = X\cos\omega_{\beta 2}t + \frac{F_0(s)}{\omega_{\beta 2}^2 - \omega_{\beta 1}^2}\left(\cos\omega_{\beta 1}t - \cos\omega_{\beta 2}t\right)$$

$$F_0(s) = \frac{e^2 NX}{2(\gamma m)l_c a}\frac{w_\perp(s)}{x_1/a}$$

- If we can arrange parameters such that the second term evaluates to zero then we will have effectively eliminated the source of BBU.

- The cosine factors in parameters represent a beat wave:
$\cos\omega_{\beta 1}t - \cos\omega_2 t = 2\sin[(\omega_{\beta 1} + \omega_{\beta 2})t/2]\sin[(\omega_{\beta 1} - \omega_{\beta 2})t/2]$



Beat wave pattern describing the amplitude of the tail relative to the head for non-equal betatron frequencies

- In practise the frequency dependence is small. The amplitude of the oscillation is given by:

$$A(s) = 2X\left(\frac{F_0(s)}{\left(\omega_{\beta 2}^2 - \omega_{\beta 1}^2\right)X} - 1\right)$$

- We specify $\omega_{\beta2} > \omega_{\beta1}$ and prescribe the condition:

$$A(s) = 2X\left(\frac{F_0(s)}{\left(\omega_{\beta2}^2 - \omega_{\beta1}^2\right)X} - 1\right)$$

- Now, $\omega_{\beta2} - \omega_{\beta1} = \Delta\omega_\beta$ such that $\omega_{\beta2}/\Delta\omega_\beta <$ and thus $\omega_{\beta2} - \omega_{\beta1} = \Delta\omega_\beta$

$$\omega_{\beta2}^2 - \omega_{\beta1}^2 = 2\omega_{\beta1}\Delta\omega_\beta$$

- Thus we arrive at a condition in which BBU is cancelled:

$$\Delta\omega_\beta = \frac{F_0(s)}{2\omega_\beta X}$$

- Let us discuss the qualitative requirement on the phase of the RF relative to the beam that is required to achieve this condition. Magnetic lenses have the property that the focussing increases with decreasing momentum. One needs to introduce a correlation between energy and axial position along the bunch. If we choose a synchronous phase later than the crest, then particles in the tail of the bunch see a smaller accelerating field compared to that experienced by the head.
- Let us quantify these requirements on the phase demanded for a correlated energy spread which cancels the energy spread induced by the short-range wake-field.

❑The betatron frequency is inversely proportional to energy, $\omega_\beta \propto \gamma^{-1}$ and for an ultra-relativistic beam β ~1 and remains essentially constant. Thus, the betatron frequency difference is given by:

$$\Delta\omega_\beta \simeq -\frac{A}{\gamma^2}\Delta\gamma \quad \text{and } \omega_{\beta 1} = \frac{A}{\gamma}$$

$$=> \quad \frac{\Delta\omega_\beta}{\omega_{\beta 1}} \simeq -\frac{\Delta\gamma}{\gamma} \quad \text{or } \Delta W_\beta \simeq -\frac{\Delta\omega_\beta}{\omega_{\beta 1}}\gamma mc^2 (= mc^2[\gamma_2 - \gamma])$$

$$\Delta W \simeq -\frac{e^2 N X}{2\frac{\omega_{\beta 1}^2}{c^2} l_c a}\frac{W_\perp}{x_1 / a}$$

In terms of the average energy $\bar{W}$:

$$\frac{\Delta W}{\bar{W}} \simeq -\frac{e^2 N X}{2\bar{W} l_c a\left(\frac{\omega_{\beta 1}}{c}\right)^2}\frac{W_\perp}{x_1 / a}$$

Finally, we address the question of the phase required to achieve the correlated energy spread. In order to do this, we calculate the energy gain at L:

$W(L) = eEL\cos\phi_1$

and the energy difference along the bunch:

$$\Delta W(L)=-eEL\sin\phi_1\Delta\phi = -eEL\sin\phi_1(\frac{L\omega s}{c}) \quad \text{(where } \Delta\phi=\omega s/c\text{).}$$

Thus, $\frac{\Delta W}{W} = -\tan\phi_1 \frac{\omega s}{c}$ and we obtain:

$$\boxed{\tan\phi_1 = -\frac{\Delta W}{W}(\frac{c}{\omega s}) \simeq \frac{e^2 Nc/(\omega s)}{4\bar{W} l_c a\left(\omega_{\beta 1}/c\right)^2} \cdot \frac{W_\perp}{x_1/a}}$$

where we have used $\Delta W/W \sim \Delta W/\bar{W}$

- The RHS of the above equation is positive and thus ϕ_1 is also required to be positive. This means the bunch will arrive later than the crest of the RF voltage (as predicted by the qualitative discussion).

- Thus we arrive at the condition $\phi_1 > 0$ which conflicts with the requirement of the longitudinal wakefield beam loading compensation condition $\phi_1 < 0$.

- Clearly, both of these conditions cannot be satisfied simultaneously!

- At SLAC the we compensate for both effects by performing a two-stage process:

1. BNS damp at the low-energy (upstream) end of the linac by shifting the phase of the klystron RF source

2. Towards the high energy (downstream) end of the linac the energy spread is compensated by changing the phase accordingly.

- **These phase changes are of the order of 15 to 20 degrees. The final energy spread of the beam is no more than 0.3 %.**

- **There is also dispersion in each accelerating structure –due to the fact that there is a frequency dependence on propagation wavenumber in any accelerating structure. This can be alleviated by tapering the profile of the RF pulse from the klystron source (see R.M. Jones et al., SLAC-PUB 10557)**

Long-Range Wake-Fields and Multi-bunch BBU

- Here, we will largely defer the discussion to the NLC case study.

- We will note the general features of the long-range wake-fields.

- The transverse wake-field which disrupts the bunch train is neither a TE or TM mode. Each mode is a hybrid HEM mode in which the ratio of TM to TE is intimately connected to the geometry of the cavity.

- In general, no more than a handful of modes will cause the beam to be significantly deflected. In other words, one has to carefully sift through the loss factors (and corresponding kick factors) of the modes to differentiate between the dominant modes and the benign modes. Also, for small displacements the dipole (E_r proportional to radial offset) mode dominates the process and will drive the beam into larger oscillation amplitudes.

- There are two main physical effects:

<u>Regenerative BBU</u>. Here the beam is deflected and it amplitude of oscillation grows within a single accelerating structure (consisting of several cells). The deflecting mode resulting from the beam-cavity interaction is carried from one cell to the next due to strong inter-cell coupling. If we consider a beam entering the cavity with a finite offset it will be deflected by the dipole magnetic field. As it transits the cavity it will loose energy to the longitudinal electric field and hence excite more fields within the cavity. This is a regenerative process, reinforced by further bunches in the train (ILC has 2,820 bunches spaced from their neighbours by 337 ns). If the field is not properly damped it will continue to grow and disrupt trailing bunches in the train. If the beam-induced growth rate exceeds the losses and once the threshold has been reached the beam will be lost to the walls of the cavity.

2. <u>Cumulative BBU</u> occurs when the beam is deflected progressively from one cavity to the next. The HEM mode excited in one cavity by an offset beam (or missaligned cells within the cavity) is then excited in the neighbouring cavity with a larger amplitude and thus a progressive build-up of fields occurs down the linac. This effect is a function of the length or number of cavities and the duration or number of bunches.

Analysis of Short-Range Wake-Fields

- In order to calculate the short-range wake-field with sufficient accuracy to faithfully represent the field a time domain approach can be used with a sufficiently fine mesh.

- However, for short bunches the mesh required is very fine indeed and this can give rise to unreasonably long computational times.

- These computational constraints are loosened considerably by using a computer code which relies on a mesh which moves with the beam and dispense with the long-range wake-field component. This method is implemented in the code Echo2D, for example.

- Similar practical limitations apply in the frequency domain. Namely, for short bunches an excessively large number of modes is required in order to properly represent the wakefield. Thus, the modal summation, as it stands, becomes impractical from the point of view of how many loss factors and synchronous frequencies that can be calculated in a reasonable amount of time and with sufficient accuracy to properly represent the wake-field.

- In general the limits this modal summation to no more than several hundred modes.

- For example, in the SLAC linac (l_c = 3.5 cm, N_{cell} = 85, a = 1.165 cm, f = 2.856 GHz, b = 4.13 cm, N_{st}= 960) 450 modes will provide an adequate representation up to 0.5 mm or so.

- Fortunately, there is a means to extend the modal calculation to shorter distances along the bunch. The method proceeds by adding to the discrete sum of modes, an integral representation of modes. This is an analytic continuation of the sum:

$$W_z(s)=2\sum_{n=1}^{N} k_n \cos(\omega_n s/c)+\Delta W_z(s)$$

$$\Delta W_z(s)=2\int_{\omega_n}^{\infty}\frac{dk}{d\omega}\cos(\omega s/c)d\omega$$

Here, we associate the impedance with:

$$Z_R(\omega)=\pi\frac{dk}{d\omega}$$

There are two regimes for this impedance:

$$Z_R(\omega)=\begin{cases} A\omega^{-3/2} & \forall\ \omega \ll cN_{cell}^{2/3}d/a^2 \\ A'\omega^{-1/2} & \forall\ \omega \gg cN_{cell}d/a^2 \end{cases}$$

a = iris radius, N_{cell} = number of cells/cavity, d = period.

This enables the analytical continuation to be obtained:

$$\Delta W_z(s) = \begin{cases} \dfrac{4A}{\pi\omega_N^{1/2}}\left\{\cos(\omega_n s/c) - \sqrt{\pi\omega_n s/c}\left[1 - 2S\left(\sqrt{\pi\omega_n s/c}\right)\right]\right\} \\ A'\sqrt{\dfrac{2c}{\pi s}}\left\{1 - C\left(\sqrt{\pi\omega_n s/c}\right)\right\} \end{cases}$$

where C and S are Fresnel integrals:

$$C(x) = \int_0^x \cos(t^2)\,dt$$

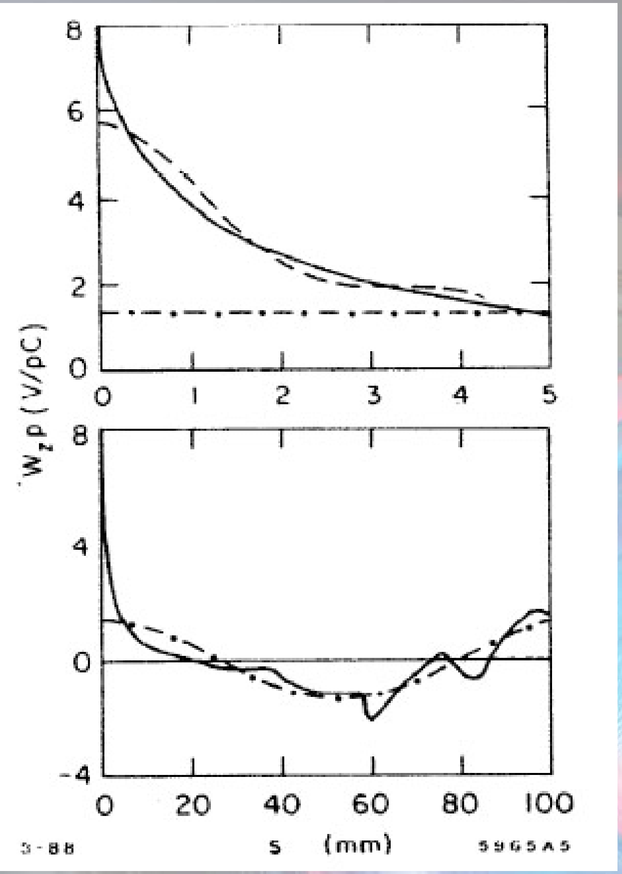
$$S(x) = \int_0^x \sin(t^2)\,dt$$

Also, A (and A') is in practise obtained as a fitting parameter.

The analytic extension is added to the modal sum until ripples (resulting from the finite sum) in the amplitude of the wake-field are damped down. This is despite the fact that the full calculation gives:

$$A = Z_R(\omega_N)\omega_N^{3/2} \text{ and } A' = \frac{Z_0}{2\pi^{3/2}}\sqrt{\frac{cg}{a^2}}.$$

Similar, formulae are obtained for the transverse wakes



Longitudinal wakefield for SLAC linac. The solid curve is the total wakefield (including analytic continuation), dashed is a discrete sum of 450 modes, dot-dashed is the fundamental accelerating mode. The long-range wake-field is well-approximated by the fundamental mode. This mode is important to consider when beam loading is evaluated.

Refs.

1.Bane & Wilson, proc. 11th conf. on High Energy Accelerators, Geneva, 1980, p581-585.

2. Heifets & Kheifets, 1988, SLAC-PUB 4625.

3. Brandt & Zotter, 1982, LEP note 388.