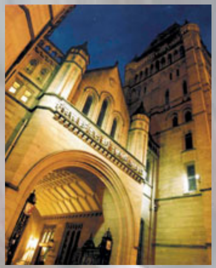
Linear Accelerators: Theory and Practical Applications: WEEK 3

Stanford Linear Accelerator, shown in an aerial digital image. The two roads seen near the accelerator are California Interstate 280 (to the East) and Sand Hill Road (along the Northwest). Image data acquired 2004-02-27 by the United States Geological Survey



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Summary of Week 2

- RF Linear accelerators for electron acceleration. Electrons have a rest mass of ~0.5 MeV and hence they are rapidly relativistic.
- The principles of infinite periodic structures were explored.
- The basic concepts of Travelling Wave (TW) acceleration were introduced.
- Fundamental accelerator physics parameters (shunt impedance, loss factor, etc) were introduced and applied to typical accelerators.
- Scaling relations were developed for the frequency dependence of shunt impedance, Q and group velocity. These parameters are used in the initial design studies of linacs.

Overview of Week 3

- **Basic concepts of SW acceleration are introduced.**
- **Fundamental concept of phase stability for ion and relativistic electron linacs is developed.**
- **The ‘capture’ condition is derived and applied to typical linacs.**
- **Principle of transverse focussing is discussed.**
- **Coupled cavity linacs –used in medical linacs as a consequence of their extraordinary stability.**
- **Circuit models of infinite periodic structures will be explored.**
- **Dispersion relations for structures consisting of a finite number of cells – influence of end cells.**
- **Important analytical relationships between mode spacing and number of cells is developed from a circuit model and applied to two practical phase advances per cell: pi and pi/2.**
- **Circuit models of electron accelerators will be used later on in the tutorial and the accuracy of these models verified.**
- **Determination of energy velocity of the space harmonic of e.m. field in periodic structure.**
- **Parameters of room temperature and superconducting cavities are introduced: LEP and TESLA, respectively.**

2. Constant Gradient Linac

➢If one wants a uniform field and power distribution along the structure the geometry has to be modified in a way such that the energy travels slower and slower, and an increased energy build-up in the cells compensates for the losses.

➢This is obtained by narrowing the iris apertures continuously and by reducing the cell diameters correspondingly to keep the cells in tune.

➢These changes have only a small influence on the shunt impedance which therefore can be regarded as constant in a first approximation. (The final result can always be corrected for the electric field variations due to varying R'.)

We integrate, $P_d' = -\frac{dP}{dz} = \text{const.}$ and obtain:

$$P(z)=P_0 - (P_0 - P(l))\frac{z}{l} = P_0\left[1-(1-e^{-2\tau})\frac{z}{l}\right] \text{ with } \tau=\int_0^l \alpha(z)dz.$$

This linearly power flow is a consequence of the linearly varying group velocity.

Using $P=P_0 e^{-2\alpha z}$ and $\alpha=\frac{\omega}{2Q_0 v_g}$ we obtain:

$$v_g = -\frac{\omega}{Q_0}\frac{P}{dP/dz} = \frac{\omega l}{Q_0}\frac{1-(1-e^{-2\tau})z/l}{(1-e^{-2\tau})}$$

The filling time:

$$T_f = \int_0^l \frac{dz}{vg} = \frac{2Q_0}{\omega}$$ and the total stored energy at the end of the filling process are equal for CI- and CG-structures.

The accelerating field is constant and from $P_0\left[1-(1-e^{-2\tau})\frac{z}{l}\right]$

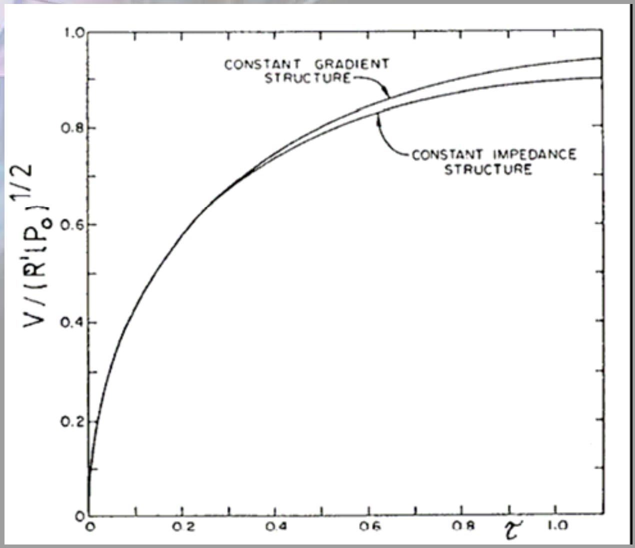
and $R'=E_{acc}^2/P_d'$:

$$E_{acc}^2 = R'P_d' = -R'\frac{dP}{dz} = R'P_0(1-e^{-2\tau})/l$$

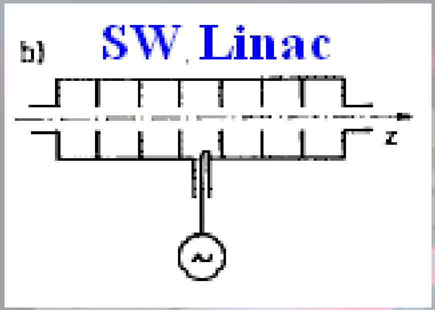
making the energy gain:

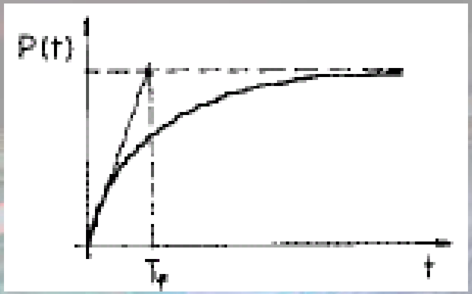
$$V=E_{acc}l = \sqrt{R'lP_0(1-e^{-2\tau})}$$

- A comparison of the energy gains of a CI- and CG-structure is illustrated adjacent. As can be seen, the gain is slightly higher in a CG-structure.
- In addition, a CG-structure is less sensitive to frequency deviations, has a higher structure efficiency and, before all, is less susceptible to beam break-up phenomena.
- In conclussion we should note that the standard high energy electron linac operates in the $2\pi/3$ TW mode in a CG-structure. For compact low energy linacs a biperiodic structure is often preferred.



3. SW Linear Acceleration





- In a SW-structure the fields build up in time.
- The incoming wave is attenuated along z, reflected at the output end and experiences another attenuation. Back at the input end the wave is partially reflected and partially transmitted through the input coupler.
- If the length of the structure is a multiple of half-wavelengths, the wave reflected at the input will be in phase with the incoming wave.
- The process of reflection at both ends will continue until an equilibrium is reached. At the equilibrium none of the backward power will pass out of the input coupler if the coupler is correctly matched. Then, the input power P_0 equals the difference between forward power P_{fo} and backward power P_{bo} at the input: $P_0 = P_{f0} - P_{b0} = (1 - e^{-4\tau})P_{f0}$

The steady state energy gain is given by:

$$V=\left[\frac{\tanh\tau/2}{\tau/2}\frac{1}{1+e^{-2\tau}}\right]^{1/2}\sqrt{R'lP_0}\cos\phi$$

and this should be compared with the traveling wave gain:

$$V=\frac{1-e^{-\tau}}{\sqrt{\tau}}\sqrt{2R'lP_0}\cos\phi,\ \tau=\alpha l.$$

Thus, the ratio of the energy gain in each accelerator is given by:

$$\frac{V_{SW}}{V_{TW}}=\left(1-e^{-4\tau}\right)^{-1/2}$$

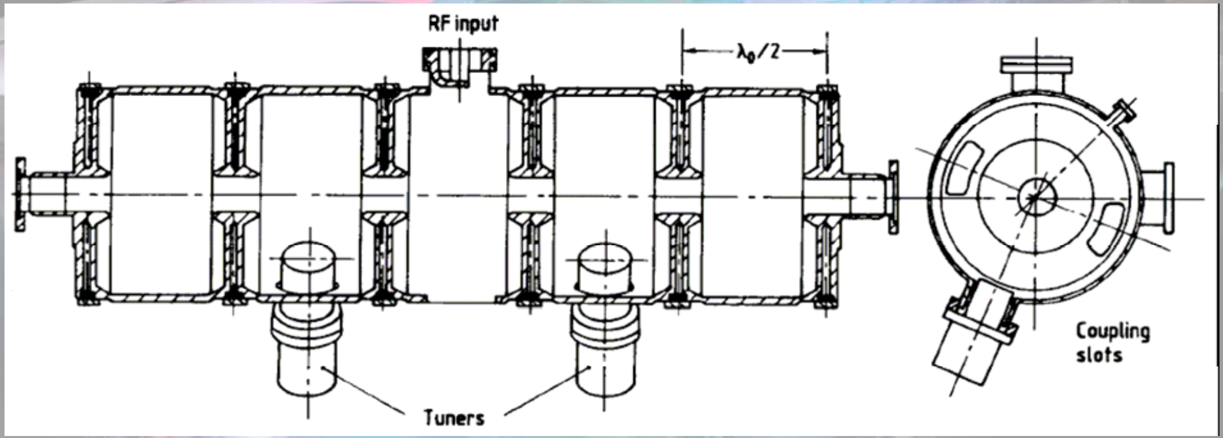
- **The improvement is larger at lower values . However, it is accomplished at the expense of an increased filling time due to multiple reflections in the structure.**
- **Other more serious disadvantages are a lower shunt impedance (only the forward wave contributes to the acceleration, while the backward wave causes additional losses) and a standing wave pattern with a peak to average field of about 2.**
- **This can be fatal in high gradient linacs because of the increased possibility of dark current generation. Dark currents consist of electrons which are emitted by field emission at "bad spots" on the structure surface**

- However, if the structure is operated in the π-mode, the forward and backward space harmonic coalesce and the fields of both add in phase resulting in twice the amplitude. In this mode the shunt impedance is as high as in a TW-structure and the peak to average field ratio is one.
- But the π-mode is related with other disadvantages. By means of a circuit model consisting of a chain of N resonators we find that the mode spacing is π /N times smaller than for the π/2 mode. Therefore, the number of cells which can be coupled is severely limited.
- Furthermore, a closer inspection shows that the π-mode has a cell-to-cell phase error and is quite sensitive to dimensional and frequency errors.-mode SW cavities for electron acceleration typically have 4 to 10 coupled cells only.
- In a normal conducting version they are used in circular accelerators at 350 to 500 MHz. At such a low frequency the iris aperture would have to be very large in order to get a reasonable coupling between cells and consequently the R/Q would be low.
- Therefore, one couples via slots in the iris wall at a position of maximum magnetic field.

➢Also, so called nose cones are added to the irises which reduce the transit time effect and concentrate the electric field at the axis.

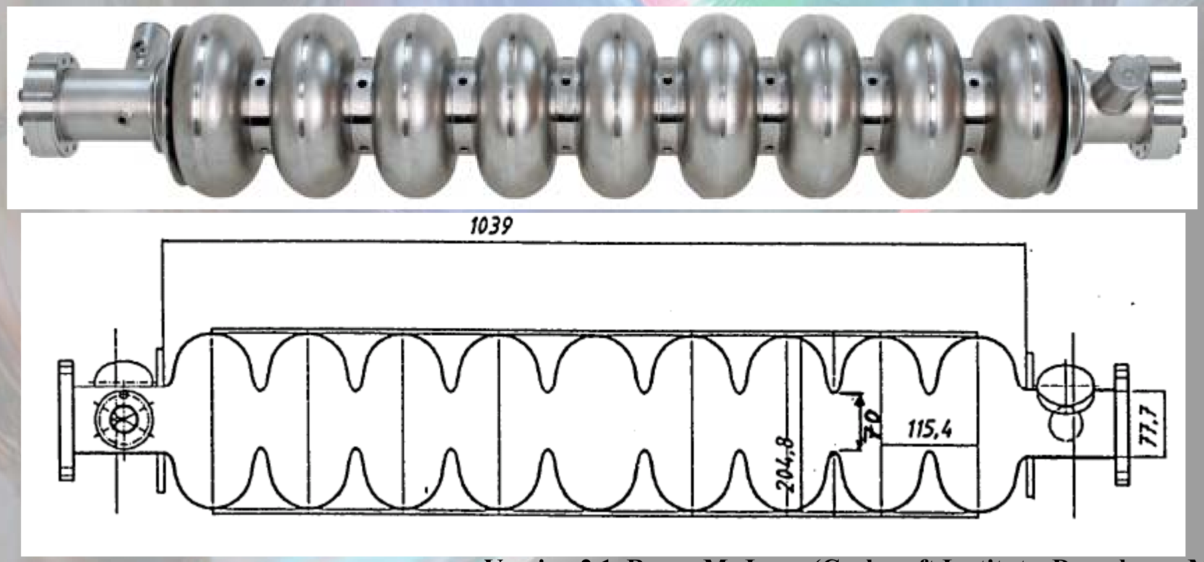
➢Finally, the cavity shape is rounded which minimizes the surface to volume ratio, that means the losses are minimized.

➢As an example the LEP cavity



Five-cell LEP cavity with magnetic coupling slots

- **Superconducting π-mode SW cavities are used in both circular and linear accelerators. They are of simple shape, straight lines and elliptical or circular arcs, with a large iris opening.**
- **The correspondingly low R/Q is of no importance in that case because of the very high Q.**
- **The rounded shape was found by chance as the one where multi-pactoring (a local resonant electron avalanche phenomenon) is suppressed.**
- **Such a cavity was originally proposed for the ILC and it is referred to as the TESLA cavity.**
- **This cavity is now used in many accelerator designs such as ERLP (Energy Recovery Linac) in Daresbury labs, the ELBE (Electron Linac for beams with High brilliance and Low Emittance) in Rossendorf accelerator facility, FLASH/TTF2 (TESLA Test Facility) at DESY in Hamburg.**



TESLA 9-Cell Cavity operating at 1.3 GHz. For the ILC the baseline design gradient is 35 MV/m.

➢Finally we note that instead of operating a SW-cavity in the π-mode one can operate in the 0-mode with good efficiency.

➢Again, forward and backward space harmonics add up in phase in every cell and both contribute to the acceleration. Since the electromagnetic fields in adjacent cells are in phase the seperating walls can be left out without affecting the field distribution.

Shunt Impedance and Q of Pill-Box

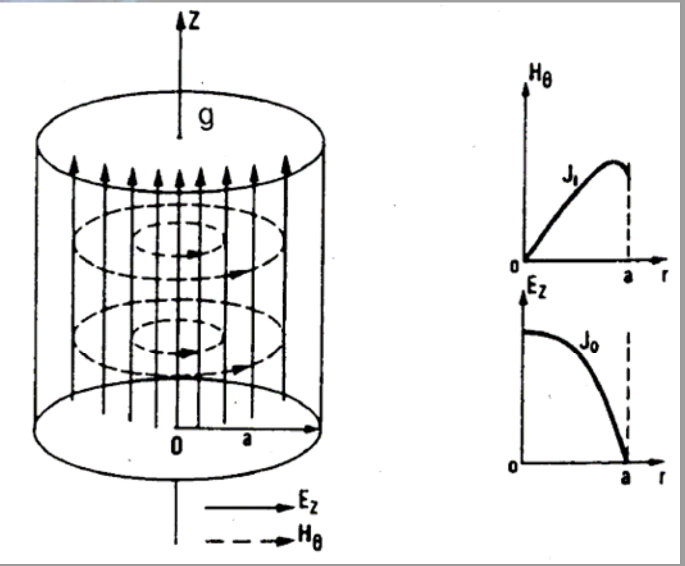
The fundamental axis-symmetric accelerating mode in a closed cylindrical cavity (pill-box) is the TM010-mode, The field components are

$$E_z = E_0 J_0\left(\chi_{01}\frac{r}{a}\right)$$

$$H_\phi = \frac{j}{Z_0} J_1\left(\chi_{01}\frac{r}{a}\right),\ Z_0 = \sqrt{\mu_0/\varepsilon_0}$$

where a is the cavity radius and χ_{01} is the first zero of the Bessel function. If, in a real cavity, the beam pipes are small as compared to the cavity the fields will not be changed appreciably and one can calculate the RF parameters in good approximation with the unperturbed fields

E.M. Field in Pill-Box Cavity



A particle which crosses the cavity on-axis will experience a voltage gain:

$$V=\int_{-g/2}^{g/2} E_z(r=0)\exp(j\omega z/c)dz = E_0 g\frac{\sin(kg/2)}{kg/2}$$

Also, the energy stored in the cavity is given by:

$$U = U_H + U_E = 2\frac{\varepsilon_0}{4}\int_0^a rdr\int_0^{2\pi} d\phi\int_{-g/2}^{g/2} dz E_0^2 J_0^2\left(\chi_{01}\frac{r}{a}\right)$$

$$=\frac{\varepsilon_0}{2}2\pi g a^2 E_0^2\int_0^1 xJ_0^2(\chi_{01}x)dx$$

$$=\frac{\pi}{2}\varepsilon_0 E_0^2 g a^2 J_1^2(\chi_{01})$$

The losses are calculated by means of the power-loss method, that is from the wall currents of the ideal conducting cavity:

$$P_d = \frac{1}{2\sigma\delta}\left\{2\int_0^a \left|H_\phi(z=0)\right| 2\pi r dr + \int_{-g/2}^{g/2}\left|H_\phi(r=a)\right|^2 2\pi a dz\right\}$$

$$=\frac{\pi}{\sigma\delta}\frac{\varepsilon_0}{\mu_0}E_0^2\left\{2a^2\int_0^1 J_1^2(\chi_{01}x)dx + agJ_1^2(\chi_{01})\right\}$$

$$\Rightarrow P_d = \frac{\pi}{\sigma\delta}\left(\frac{E_0}{Z_0}\right)^2 a(a+g)J_1^2(\chi_{01})$$

where σ the conductivity of the metal and the skin depth is given by: $\delta = 2/\sqrt{\omega\mu_0\sigma}$.

The shunt impedance is given by:

$$R = \frac{V^2}{P_d} = \frac{1}{\pi}\sigma\delta Z_0^2 \frac{(g/a)^2}{1+g/a}\left(\frac{\sin(kg/2)}{kg/2}\right)^2 \frac{1}{J_1^2(\chi_{01})}$$

and the quality factor:

$$Q = \frac{\omega U}{P_d} = \frac{1}{\delta}\frac{g}{1+g/a}$$

and for the sake of completeness:

$$R/Q = \frac{1}{\pi}\sigma\delta^2 Z_0^2 \frac{g}{a^2}\left(\frac{\sin(kg/2)}{kg/2}\right)^2 \frac{1}{J_1^2(\chi_{01})}$$

and since it is a TM_{010} mode the frequency is given by: $\omega/2\pi = \chi_{01}c/(2\pi a)$

For example a Cu cavity at $\omega/2\pi$=12 GHz of length $\lambda/3$ ($2\pi/3$ mode):

$$\chi_{01} = 2.405,\ J_1(\chi_{01}) = 0.5191,\ \sigma = 5.8\text{x}10^7\,\Omega^{-1}\text{m}^{-1}$$

$$a = \frac{\chi_{01} c}{\omega} = 9.57\text{ mm},\ \ g = \frac{\lambda}{3} = 8.33\text{ mm},\ \delta = 0.85\ \mu\text{m}$$

$$T = \frac{\sin kg/2}{kg/2} = \frac{3\sqrt{3}}{2\pi} \sim 0.827$$

$$R' = R/g = 276\ \text{M}\Omega\text{m}^{-1},\ Q \sim 5221$$

$$R'/Q \sim 53\ \text{k}\Omega\text{m}^{-1}$$

and the filling time: $T_0 = 2Q/\omega = 0.14\ \mu\text{s}$.

Compare this to a 3 GHz cavity of length $\lambda/2$:

$$a = \frac{\chi_{01} c}{\omega} = 38.3\text{ mm},\ \ g = \frac{\lambda}{3} = 50.0\text{ mm},\ \delta = 1.71\ \mu\text{m}$$

$$T = \frac{\sin kg/2}{kg/2} = \frac{2}{\pi} \sim 0.64$$

$$R' = R/g = 100\ \text{M}\Omega\text{m}^{-1},\ Q \sim 12{,}706$$

$$R'/Q \sim 8\ \text{k}\Omega\text{m}^{-1}$$

and the filling time: $T_0 = 2Q/\omega = 1.35\ \mu\text{s}$

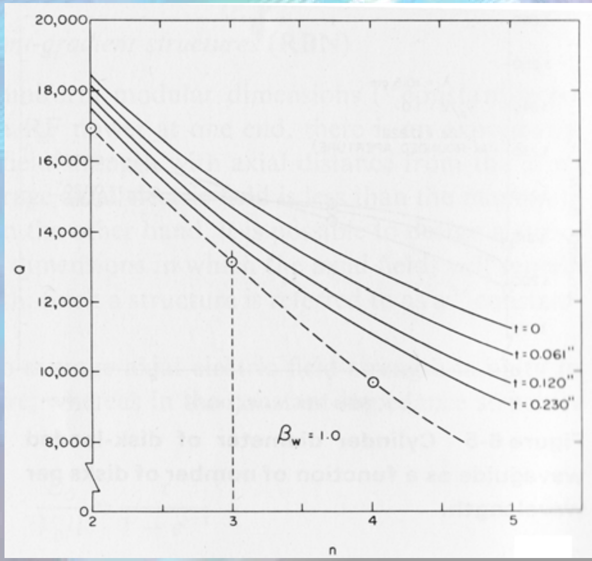
If we replace the gap, g, with the period - the iris thickness then we get an approximate formula for the shunt impedance of a periodic structure:

$$R' = 968 \frac{(v_p/c)}{\delta} \frac{(1-\eta)^2}{n + 2.61(v_p/c)(1-\eta)} \left(\frac{\sin(D/2)}{D/2} \right)^2$$

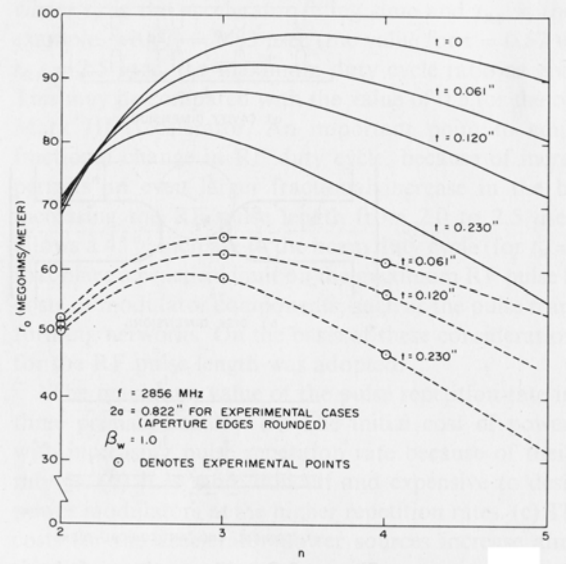
where the phase velocity is not assumed to be c, d is the period of the structure, t is the disk thickness, η is fraction of the length occupied by disks (=t/d=tn/λ), n is the number of disks per wavelength and, D is the transit angle of an electron passing through the gap ($= \frac{2\pi}{\lambda}(d-t)$).

Similarly, the Q is given by:

$$Q = \frac{\lambda}{\delta} \frac{(v_p/c)(1-\eta)}{n + 2.61(v_p/c)(1-\eta)}$$



Q-Factor versus number of irises per wavelength for various thicknesses. Experimental values are indicated with the dashed curve.



Shunt Impedance versus number of irises per wavelength for various thicknesses. Experimental values are indicated with the dashed curve

Ref: Chapt. 6, G. A. Loew et al , "The Stanford Two-Mile Accelerator", W.A. Benjamin (1968) ed. Richard Neal.

Energy Velocity

For In order to calculate the energy velocity we focus on the fundamental space harmonic:

$$E_z = E_0 J_0(k_{c,0} r)\exp[j(\omega t - \beta_0 z)],\ \beta_0^2 = \left(\frac{\omega}{c}\right)^2 - k_{c,0}^2$$

$$E_r = jE_0 \frac{\beta_0}{k_{c,0}} J_1(k_{c,0} r)\exp[j(\omega t - \beta_0 z)]$$

$$H_\phi = jE_0 \frac{\omega\varepsilon}{k_{c,0}} J_1(k_{c,0} r)\exp[j(\omega t - \beta_0 z)]$$

The power flow is given by:

$$P = \frac{1}{2}\int_0^a E_r H_\phi^* 2\pi r dr = \pi\beta_0\omega\varepsilon_0 \frac{E_0^2}{k_{c,0}^2}\int_0^a r J_1^2(k_{c,0} r) dr$$

and the stored energy per unit length:

$$U' = \frac{\mu_0}{2}\int_0^a \left|H_\phi^2\right| 2\pi r dr = \pi\mu_0\omega^2\varepsilon_0^2 \frac{E_0^2}{k_{c,0}^2}\int_0^a r J_1^2(k_{c,0} r) dr$$

.

The velocity at which the energy is transported is given by:

$$v_E = \frac{P}{U'} = \frac{\beta_0}{\omega \mu_0 \varepsilon_0} = \beta_0 \frac{c^2}{\omega}$$

on the other hand from the definition of the group velocity:

$$v_g = \frac{\partial \omega}{\partial \beta_0} = \frac{1}{\frac{\partial \beta_0}{\partial \omega}} = \frac{c^2}{\omega}\sqrt{\left(\frac{\omega}{c}\right)^2 - k_{c,0}^2} = \frac{c^2}{\omega}\beta_0$$

Thus, we find the energy velocity is equal to the group velocity:

$$v_E = v_g$$

RF Linac Dynamics of Electron Capture

- Here we consider the import concept of "electron capture"
- Low-energy electrons are injected into v=c electron linac (designed assuming the electrons are ultra-relativisitic and hence assumes the length of each gap and iris is constant as the velocity is constant)
- This means the electrons will slip in phase with respect to the accelerating field.

The equation of motion of a paricle at z,t accelerated by

an electric field wuth amplitude E_0:

$$\frac{d}{dt} mc\beta\gamma = qE_0 \cos\phi(z,t)$$

where the phase of the traveling wave

$$\phi(z,t) = \omega t - \frac{2\pi z}{\lambda}$$

Differentiating wrt time t we obtain the

equation of phase motion is described by:

$$\frac{d\phi}{dt} = \frac{2\pi c}{\lambda}(1-\beta)$$

Now, as $\beta<1$ then ϕ increases with time and the particle falls further and further behind the initial phase of the wave. Changing variables from time to phase in the equation of motion:

$$mc\gamma^3 \frac{d\beta}{d\phi}\frac{d\phi}{dt} = qE_0 \cos\phi,$$ where we have used $d(\gamma\beta)=\gamma^3 d\beta$.

The equation of phase is also written:

$$\frac{1}{1+\beta\sqrt{1-\beta^2}}\frac{d\beta}{d\phi} = \frac{qE_0\lambda}{2\pi mc^2}\cos\phi$$

This can now be integrated to obtain the dependence of phase on velocity during the process of acceleration. In order to facilitate this process we change variables : $\beta=\cos\alpha$ and we utilise the integral result:

$$\int\frac{d\alpha}{1+\cos\alpha} = \tan\frac{\alpha}{2}$$ and the trigonometric identity:

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \sqrt{\frac{1-\beta}{1+\beta}}$$

and we obtain:

$$\sin\phi=\sin\phi_i + \frac{2\pi mc^2}{qE_0\lambda}\left(\sqrt{\frac{1-\beta_i}{1+\beta_i}} - \sqrt{\frac{1-\beta}{1+\beta}}\right)$$

➢ This indicates that there is no oscillatory phase motion about the synchronous phase.

➢ Also, since $\beta > \beta_I$ the term in parenthasis continues to increase (until β=1) and the phase becomes more and more positive as β increases.

➢ To ensure the particles approach the crest (maximum acceleration) the particle are required to be injected with a negative phase.

➢ Eventually the phase approaches an asymptotic value:

$$\sin\phi_\infty = \sin\phi_i + \frac{2\pi mc^2}{qE_0\lambda}\sqrt{\frac{1-\beta_i}{1+\beta_i}}$$

and we define the second term as:

$$F = \frac{2\pi mc^2}{qE_0\lambda}\sqrt{\frac{1-\beta_i}{1+\beta_i}}$$

The phase slip is maximised for a minimum electric field.

Let us choose $\phi_i \sim -90,$ and a little more positive to ensure acceleration.

If we choose F=1 then this corresponds to the desired effect, namely, to placing the asymptotic phase on the peak of the accelerating wave. This implies the field is required to be:

$$E_0 = \frac{2\pi mc^2}{q\lambda}\sqrt{\frac{1-\beta_i}{1+\beta_i}}$$

- ➢The practical implications of this is that a low-injection velocity requires a large electric field gradient.
- ➢During the capture process, as the injected electron beam moves up the crest, the beam becomes bunched -due to velocity modulation as the earlier particles experience a lower electric field than the later ones.

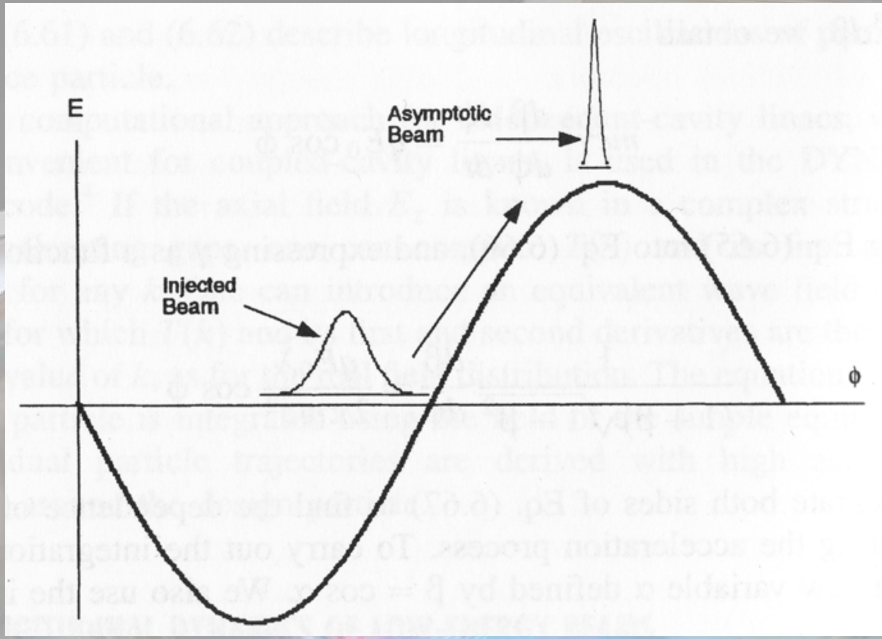
As a first order estimate of the effect we again consider $\phi_\infty \sim 0$ and pressume the initial phase width is $2\delta\phi_i$ and the intial centroid at $-\pi/2+\delta\phi_i$. This makes $F = 1 - \delta\phi_i^2/2$ and the particle with the earliest initial phase will have an asymptotic phase $\phi_\infty = -\delta\phi_i^2/2$ which is just earlier than the crest.

On the other hand, a particle with the latest initial phase will be asymptotically at $\phi_\infty = 3\delta\phi_i^2/2$.

This final half-width of the beam is $\delta\phi_\infty = \delta\phi_i^2$

- ➢In summary, if low-velocity electrons are injected into an accelerating structure whose phase velocity is that of the velocity of light, they slip in phase but if the accelerating phase is chosen correctly, the particles approach the crest asymptotically where they can be efficiently accelerated to high energies.
- ➢The capture process can also result in significant bunching of the beam.

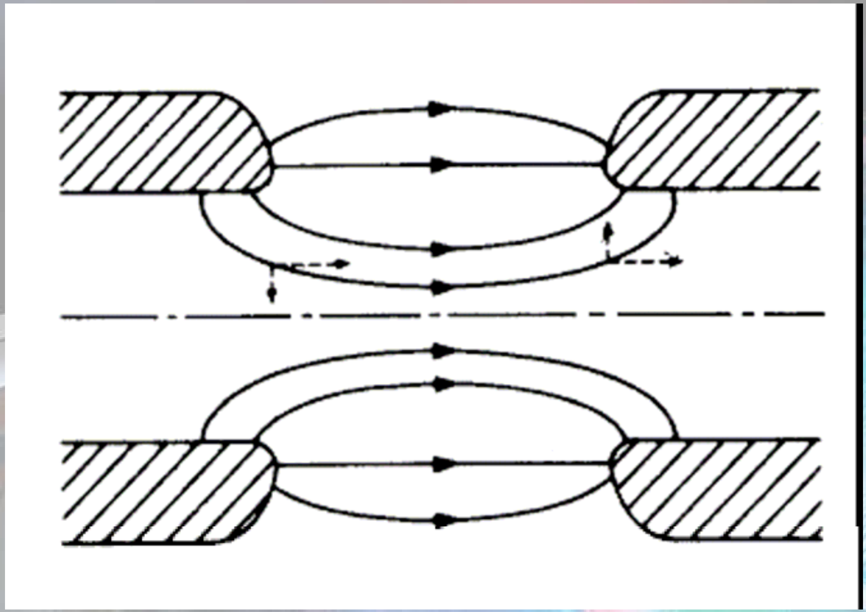
- For example, consider injection of 1 MeV electrons into a traveling wave structure based on a 2 GHz ($\omega = 1.26 \times 10^{10}\ s^{-1}$) iris loaded waveguide.
- The quantity $\beta_0 = 0.9411$ and to be captured on the crest this makes $E_0 \sim 3.7$ MV/m –this is quite realistic.



Longitudinal dynamics of a low-energy electron beam injected into a v=c linac accelerator

- For example, consider injection of 150 keV electrons into a traveling wave structure based on a 3 GHz ($\lambda = 10$ cm, $\omega = 1.88 \times 10^{10}\ s^{-1}$) iris loaded waveguide.
- The capture condition makes $E_0 > 7.6$ MV/m –this is technically possible.
- In practise a small bunching section is used to bring it to within 1 MeV.

Transverse Focussing



Electric field distribution between drift tubes in an Alvarez DTL

- In the gap between drift tubes there is an electric field
- This field is focussing at the gap entrance and defocussing at the exit.
- In an electrostatic accelerator where the field is constant this gives an overall focussing effect since the particles having more energy at the end of the gap make the defocusing effect smaller.

- In an RF accelerator the opposite occurs. From the perspective of the phase stability requirement ($\phi_s < 0$) it appears that the field increases with time during the passage of the particle.
- Hence the defocussing term becomes larger than the focussing one, resulting in a transverse instability which may cause the particles to impact upon the drift tubes.

To illustrate this effect we consider small offsets and obtain the fields:

$$E_z = E_0 \sin\left(\omega t - \int \frac{dz}{v_p}\right)$$

$$E_r = \frac{r\omega E_0}{2v_p} \cos\left(\omega t - \int \frac{dz}{v_p}\right)$$

$$B_\phi = \frac{r\omega E_0}{2c^2} \cos\left(\omega t - \int \frac{dz}{v_p}\right)$$

The transverse force deflecting the particle is given by the Lorentz force equation:

$$\frac{d}{dt} m\dot{r} = eE_r - evB_\theta$$

$$= \frac{er\omega E_0}{2v_p}\left(1 - \frac{vv_p}{c^2}\right) \cos\left(\omega t - \int \frac{dz}{v_p}\right)$$

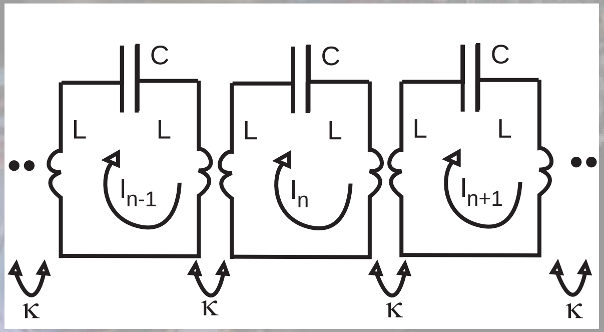
If we consider the synchronous particle for which $v=v_p$ and $\omega t - \int \frac{dz}{v_p} = \phi_s + \frac{\pi}{2}$:

$$\frac{d}{dt} m\dot{r} = -\frac{er\omega E_0}{2v_p}(1-\beta^2)\sin\phi_s$$

- **For ultra-relativistic particles this defocussing force is negligible. The electric force is compensated by the magnetic force.**
- **There are other reasons for deflection of particles other than the normal transverse field components in a waveguide.**
- **The fundamental mode couplers introduce an asymmetry in fields and this can cause significant transverse deflection of the beam –in the TESLA accelerator for the ILC there is a significant quadrupole kick from the fundamental mode couplers.**
- **This is the reason such as solenoids or magnetic quadrupoles are required to focus the beam in practise and to ensure a stable transverse motion.**

Monopole Mode Dispersion Relations

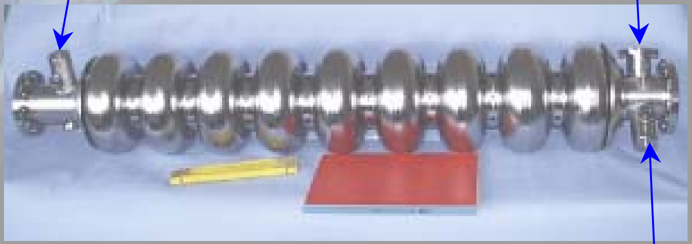
- Utilize an L-C circuit model.
- This represent an infinitely repeating periodic structure.
- For our purposes the Ls and Cs are the same for each cell in the coupled chain.
- A mutual inductance is used to represent cell-to-cell coupling. The sign of this coupling changes the group velocity of the mode (i.e. TE mode coupling differs from TM).
- Generalizing this method (with appropriate boundary conditions) allows the tuning characteristics to be obtained –field flatness, resonant frequency.



L-C Circuit Model

HOM Coupler

Fundamental I/P Coupler



HOM Coupler

9-Cell Niobium Cavity Suitable for ILC

- Circuit model coupled equations:

$$\left(2j\omega L + \frac{1}{j\omega C}\right) i_n + j\omega M (i_{n+1} + i_{n-1}) = 0$$

Here M =κL is the mutual coupling of neighbouring cells.

Dividing by $2j\omega L$ gives:

$$\left(1-\frac{\omega_r^2}{\omega^2}\right)i_n+\frac{\kappa}{2}(i_{n+1}+i_{n-1})=0, \quad \text{where } \omega_r=\frac{1}{\sqrt{2LC}}.$$

For an infinitely periodic structure the phase changes across one cell by ϕ.

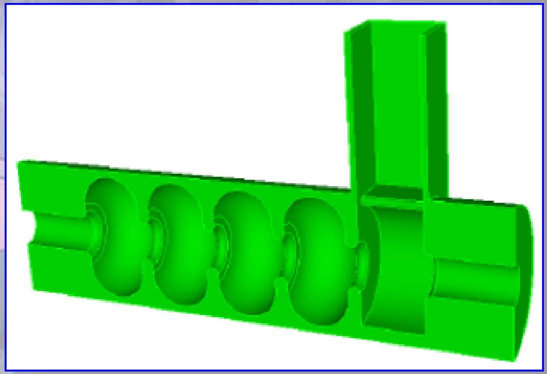
$$\Rightarrow i_n = i_0 e^{jn\phi}$$

$$\left(1-\frac{\omega_r^2}{\omega^2}\right)e^{jn\phi}+\frac{\kappa}{2}(e^{j\phi}+e^{-j\phi})e^{jn\phi}=0$$

$$\left(1-\frac{\omega_r^2}{\omega^2}\right)+\kappa\text{Cos}\phi=0$$

$$\Rightarrow \omega=\frac{\omega_{\pi/2}}{\sqrt{1+\kappa\text{Cos}\phi}}$$

where the $\omega_{\pi/2}$ is recognised as the cell resonant frequency (verify by letting $\phi=\pi/2$).



CAD Model of 5-Cell Positron Capture Cavity
(designed and fabricated at SLAC)

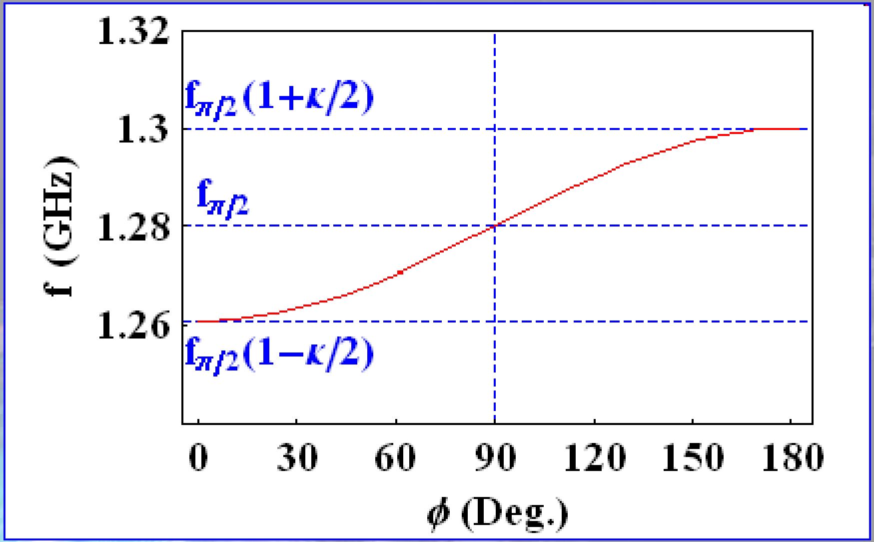
In general $\kappa<<1$

$$\Rightarrow \omega \sim \omega_{\pi/2}\left(1-\frac{\kappa}{2}\text{Cos}\phi\right)$$

$\Rightarrow \omega_0 \sim \omega_{\pi/2}\left(1-\kappa/2\right)$ and $\omega_\pi \sim \omega_{\pi/2}\left(1+\kappa/2\right)$.

Thus the bandwidth of the cell defines the coupling coefficient:

$$\kappa = \frac{\left(\omega_\pi - \omega_0\right)}{\omega_{\pi/2}}.$$



Dispersion Curve for Infinitely Periodic Structure

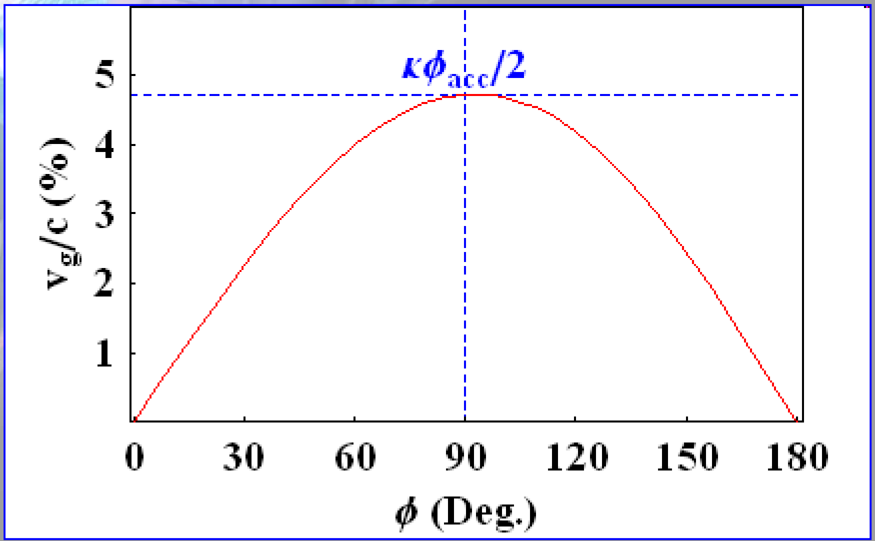
(κ~3.05%, $f_{\pi/2}$~1.28 GHz)

Group velocity:

$$v_g/c = \frac{p}{c}\frac{d\omega}{d\phi} = p\frac{\omega_{\pi/2}\kappa \text{Sin}\phi}{2c}. \text{ Using } p = \frac{c\phi_{acc}}{\omega_{acc}}$$

$$\Rightarrow \frac{\phi_{acc}}{2\omega_{\pi/2}\left(1-\frac{\kappa}{2}\text{Cos}\phi_{acc}\right)}\omega_{\pi/2}\kappa \text{Sin}\phi$$

$$\boxed{v_g/c = \frac{\phi_{acc}\kappa \text{Sin}\phi}{2\left(1-\frac{\kappa}{2}\text{Cos}\phi_{acc}\right)} \sim \frac{\phi_{acc}\kappa \text{Sin}\phi}{2}}$$

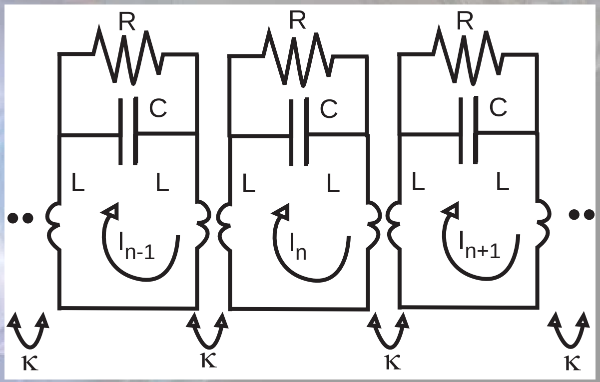


Group Velocity

(Corresponding to above dispersion parameters.)

Dispersion Relations for Cavities with Finite Qs

- To add losses we could, of course, re-analyze the whole circuit but it is is more straightforward if we analyze only the the C-R part of the circuit. The C-R impedance is given by:



$$\frac{1}{j\omega C} \rightarrow \frac{\frac{R}{j\omega C}}{R + \frac{1}{j\omega C}}$$

$$\Rightarrow \frac{1}{C} \rightarrow \frac{1}{C}\frac{1}{1 + \frac{1}{j\omega CR}} = \frac{1}{C}\frac{1}{\left(1 + \frac{\omega_r}{j\omega Q}\right)}$$

Here we have used $Q=\omega_0 CR$ for a parallel resonant cct.

Thus, $C \rightarrow C\left(1+\frac{\omega_r}{j\omega Q}\right)$ and we can modify the resonant frequency term in the dispersion relation accordingly:

$$\omega_r \rightarrow \frac{\omega_r}{\sqrt{1 + \frac{\omega_r}{j\omega Q}}} \text{ and this makes } \omega = \frac{\omega_r}{\sqrt{1+\kappa \mathrm{Cos}\phi}}\frac{1}{\sqrt{1 + \frac{\omega_r}{j\omega Q}}}.$$

Let $\omega_\kappa = \dfrac{\omega_r}{\sqrt{1+\kappa \text{Cos}\phi}}$ and square both sides of the dispersion relation

$$\Rightarrow \omega_\kappa^2 = \left(1 + \frac{\omega_r}{j\omega Q}\right)\omega^2.$$

The resulting quadratic, $\omega^2 + \dfrac{\omega\omega_r}{jQ} - \omega_k^2 = 0$, is readily solved:

$$\omega = \frac{j\omega_r}{2Q} \pm \sqrt{-\frac{\omega_r^2}{4Q^2} + \omega_\kappa^2}$$ and for large Qs this reduces to:

$$\cong \frac{j\omega_r}{2Q} + \omega_\kappa\left(1 - \frac{\omega_r^2}{8Q^2\omega_\kappa^2}\right).$$

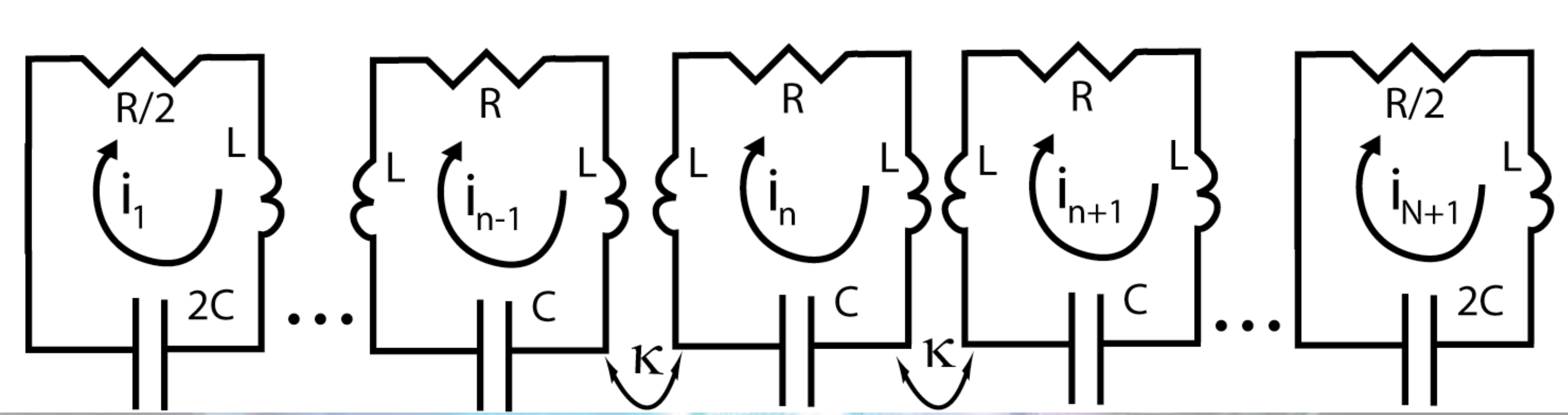
In practise Q>>1 and it is safe to drop the term in parenthesis (as it is second order in Q):

$$\boxed{\omega = \frac{\omega_{\pi/2}}{\sqrt{1+\kappa \text{Cos}\phi}} + \frac{j\omega_{\pi/2}}{2Q}}$$

The field varies as $e^{-j\omega t}$ and thus it decays as $\dfrac{\omega_{\pi/2}}{2Q}$ (as we would expect!).

Also, the mode frequency is reduced due to the losses.

Dispersion Relations for Finite-Length Linacs



N-Cell Cavity, Inductively Coupled and Terminated in Half-Cells

- **Model of a linear chain of N coupled resonators. Each resonator represents the fundamental mode resonance (TM_{010} resonance) of a pill-box cavity.**
- **The coupling between cavities, via the irises, is simulated by the mutual inductance M (= κL) of a transformer.**
- **The chain is terminated in half-cells corresponding to ideal conducting mid-planes in the end cavities. (This yields a "flat" constant current distribution, π-mode. If the chain were terminated in full cells, the end-cells would have to be detuned in order to get a flat -mode.)**

The modal quality factor is given by: $Q = \dfrac{2\omega_r L}{R}, \ \omega_r = (2LC)^{-1/2}$

and the equations for the central and end cell loops are:

$$\left(j\omega L - \frac{j}{2\omega C} + \frac{R}{2} \right) i_1 + i_2 j\omega L\kappa = 0$$

$$\left(2j\omega L - \frac{j}{\omega C} + R \right) i_n + j\omega L\kappa (i_{n+1} + i_{n-1}) = 0, \ \text{where } n=2,3,..N$$

$$\left(j\omega L - \frac{j}{2\omega C} + \frac{R}{2} \right) i_{N+1} + i_N j\omega L\kappa = 0$$

which are readily re-written as:

$$\left(1 - \frac{\omega_r^2}{\omega^2} - j\frac{\omega_r}{\omega Q} \right) i_1 + i_2 \kappa = 0$$

$$\left(1 - \frac{\omega_r^2}{\omega^2} - j\frac{\omega_r}{\omega Q} \right) i_n + \frac{\kappa}{2} (i_{n+1} + i_{n-1}) = 0$$

$$\left(1 - \frac{\omega_r^2}{\omega^2} - j\frac{\omega_r}{\omega Q} \right) i_{N+1} + i_N \kappa = 0$$

Using $\alpha = \frac{2}{\kappa}\left(1 - \frac{\omega_r^2}{\omega^2} - j\frac{\omega_r}{Q\omega}\right)$ then:

$$\frac{\alpha}{2} i_1 + i_2 = 0$$

$$\alpha i_n + \left(i_{n+1} + i_{n-1}\right) = 0$$

$$\frac{\alpha}{2} i_{N+1} + i_N = 0$$

Multiplying and subtracting we obtain an upper triangular matrix:

$$\frac{\alpha}{2} i_1 + i_2 = 0,\ (\alpha - \frac{1}{\alpha/2}) i_2 + i_3 = 0,\ (\alpha - \frac{1}{\alpha - (1/\alpha/2)}) i_3 + i_4,\ ....$$

In the diagonal element we make a substitution $\alpha = -2\cos\phi$:

$$\frac{\alpha}{2} = -\cos\phi$$

$$\alpha - \frac{1}{\frac{1}{2}\alpha} = -\frac{\cos 2\phi}{\cos\phi}$$

$$\alpha - \frac{1}{\alpha - \frac{1}{\frac{1}{2}\alpha}} = -2\cos\phi + \frac{\cos\phi}{\cos 2\phi} = -\frac{\cos 3\phi}{\cos 2\phi}$$

$$\frac{-\cos(N+1)\phi+\cos(N-1)\phi}{2\cos N\phi}=\frac{\sin\phi\sin N\phi}{\cos N\phi}$$

The determinant of a triangular matrix is equal to the product of the diagonal elements for a homogeneous system we set it equal to zero:

$\det = (-)^N \sin\phi \sin N\phi = 0$

$\Rightarrow$ Eigenvalues are given by $\phi_n = n\pi/N$ (n=0,1,2..N)

and $\alpha_n = \frac{2}{\kappa}\left(1-\frac{\omega_r^2}{\omega_n^2}-j\frac{\omega_r}{\omega_n Q}\right) = -2\cos\phi_n$

For Q>>1 we obtain:

$$\omega_n = \omega_r / \sqrt{\left[1+\kappa\cos(n\pi/N)\right]} \sim \omega_{\pi/2}\left[1-\frac{1}{2}\kappa\cos(n\pi/N)\right]$$

For a general value of Q solving for ω_n:

$$\omega_n = \omega_{\pi/2}\left(-\frac{j}{2Q}+\sqrt{-\frac{1}{4Q^2}+1+\kappa\cos(n\pi/N)}\right)^{-1}$$

The real component of frequency is increased due to the presence of losses:

$$\mathrm{Re}\{\omega_n\} \sim \omega_{\pi/2}\left[1+\frac{\kappa}{2}\cos(n\pi/N)+1/(8Q^2)\right]$$

Similarly, for the eigenvectors:

$$i_n = \cos(p-1)\phi_n = \cos\left[(p-1)n\pi / N\right]$$

Clearly, ϕ_n is the phase shift per cell. The bandwidth defines the separation between the 0 and π mode:

$$\Delta\omega = \omega_N - \omega_0 \sim \kappa\omega_{\pi/2}$$

Thus, the coupling factor, $\kappa = \Delta\omega / \omega_{\pi/2}$ (=M/L) is the normalised bandwidth.

The nearest neighbour mode separation is an important quantity.

For the π and the $\pi/2$ modes we find:

$$\frac{\Delta\omega_\pi}{\omega} = \frac{\omega_N - \omega_{N-1}}{\omega_{\pi/2}} \sim \frac{\kappa}{2}(1-\cos\frac{\pi}{N}) \sim \frac{\kappa\pi^2}{4N^2}$$

$$\frac{\Delta\omega_{\pi/2}}{\omega} = \frac{\omega_{N/2} - \omega_{(N-1)/2}}{\omega_{\pi/2}} \sim \frac{\kappa}{2}\cos\left(\frac{\pi}{2} - \frac{\pi}{2N}\right) \sim \frac{\kappa\pi}{4N}$$

Thus, the mode separation is π/N times smaller for the π mode as compared to the $\pi/2$ mode.

Now, the mode separation should be larger than that defined by the Q losses bandwidth ($=\omega_r/Q_0$):

$$N_\pi < \frac{\pi}{2}\sqrt{\kappa Q_0} \quad \text{and} \quad N_{\pi/2} < \frac{\pi}{4}\kappa Q_0$$

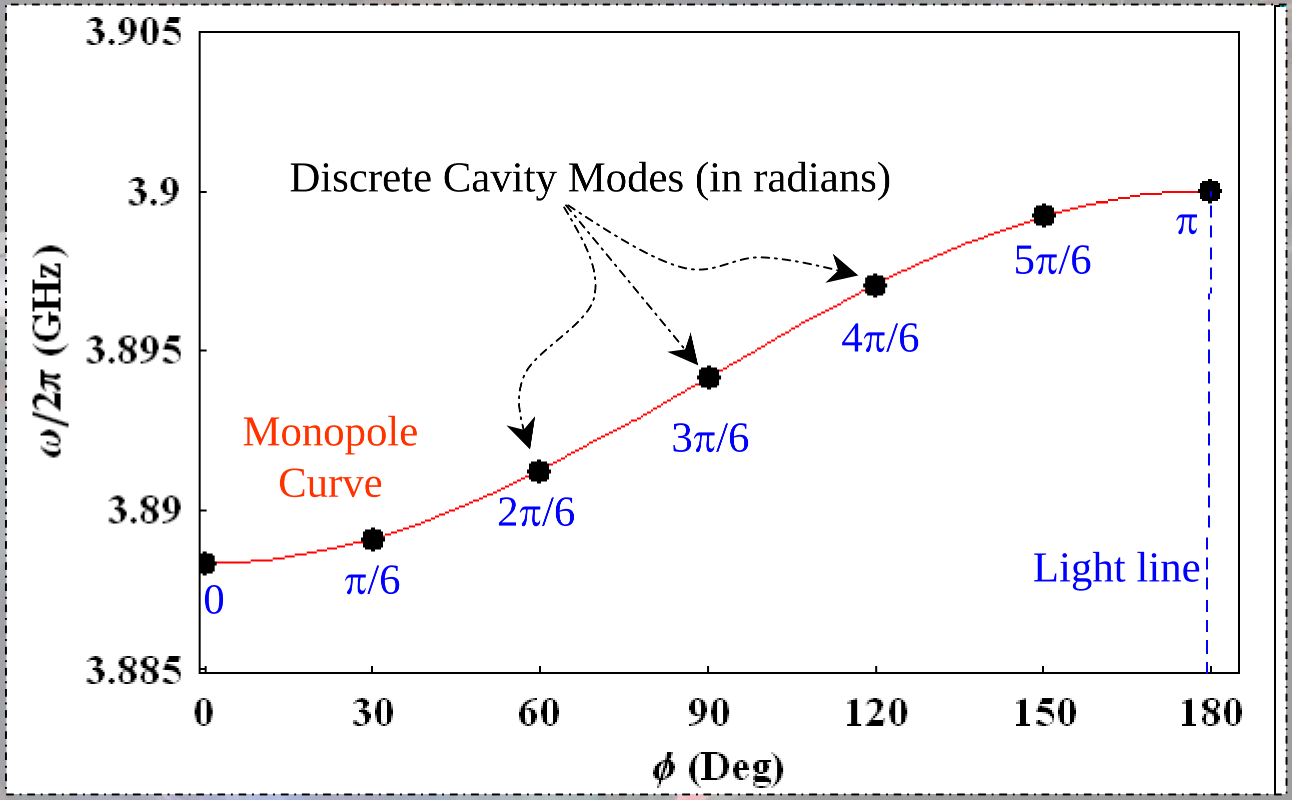
For example, for $f_0 \sim 1$ to 3 GHz , $Q \sim 10^4$ and $\kappa \sim 1\%$:

$\Rightarrow N_\pi < 15$ and $N_{\pi/2} < 78$

Thus, the number of coupled cavities is limited in the π-mode and the number of feed points must be increased correspondingly. An additional analysis of the π mode indicates that there are cell-to-cell phase errors due to Q-losses, and morever it is rather sensitive to dimensional or frequency errors. The $\pi/2$ mode, on the other hand, is far less sensitive to these effects. For these reasons, for room temperature RF linacs one often operates in the $\pi/2$ or $2\pi/3$ mode.

However, we note there is technique to mitigate against the sensivity dissadvantages of the π mode. Namely, we add coupling cavities to either side of the main accelerating cavities in a such a way that the chain uses a $\pi/2$ mode while the main-to-main cell phase shift is π.

The π mode is the preferred mode of operation from the perspective of efficiency –the ‘shunt impedance’ is maximised.



Dispersion curve for linac. Also indicated are the discrete modes for 7 coupled oscillators