Inertial Navigation: A Bridge Between Kinematics and Calculus

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Those who come to Cambridge soon learn that the fastest route between Harvard and MIT is by the subway. For many students, this short ride is a quick and easy way to link physics and calculus. A simple, home-made accelerometer provides all the instrumentation necessary to produce accurate graphs of acceleration, velocity, and displacement position on this subterranean route. Each graph, in turn, reveals a story of the train's trip: speeding up and braking, varying track conditions, top speed, and the distance between subway stops deep beneath Massachusetts Avenue.

Using a horizontal accelerometer illustrates the fundamentals of inertial guidance, a reliable and accurate means to track the movements of airplanes, submarines, missiles, and spacecraft. Unlike GPS, there is no signal to be jammed nor interfered with by terrain or weather. It cannot be otherwise disturbed from afar and requires no radio frequency reception that can be compromised by intervening water or earth. No external reference is needed once initialized with a starting position and velocity. The key sensors in commercial and military "strap down" inertial navigation systems are three orthogonally oriented accelerometers. Gyroscopic stabilization provides an inertial reference frame. However, for simple systems in which one is interested in movement along a single dimension and on a level surface, one rudimentary accelerometer is sufficient.

While electronic accelerometers that record automatically are available for microcomputers and routinely are built into newer cell phones, it is far more instructive to construct one from commonly available materials. A bubble or ball in a curved tube is responsive to horizontal acceleration. William U. Walton designed the first bubble-tube accelerometer and a means to calibrate it and incorporated it as a part of a popular introductory calculus curriculum in the 1970s. Others have "rediscovered" his invention over the ensuing 40 years, using it to measure angular acceleration, marketing versions for classroom use, or describing building versions for amusement park field trips. These simple accelerometers' tremendous potential to illustrate the principles of inertial navigation and their link to integral calculus merit their inclusion in any physics course.

Our accelerometer is built from a foot of clear flexible tubing (1/4-in i.d. works fine), two stoppers, and a single BB or stainless steel ball bearing, all commonly available items (Fig. 1). The tubing can be affixed to any convenient backing with tape or hot-melt glue and then filled with water. Any simple curve will work (we have had success with a 30° arc of a circle of 22-in [56-cm] diameter). The key to converting what looks like a rudimentary bubble level into an accelerometer is calibration.

The physics is quite simple and similar to that of a pendulum except that the path of the mass is constrained to move along a circular arc by a plastic tube rather than a taut string (Fig. 2). The relevant forces acting on a bead anywhere along a curved tube are gravity and the force due to acceleration. When the forces normal to the arc add to zero, the bead is motionless. Its position along the arc indicates the value of the acceleration at that moment. Water in the tube dampens oscillations, so the position of the bead is easy to read.

\[ F_1 = mg \cos \theta, \]
\[ F_2 = -ma \sin \theta, \]
\[ \Sigma \text{forces} = F_1 + F_2 = mg \cos \theta - ma \sin \theta = 0 \]
\[ a = g \sin \theta \cos \theta = g \tan \theta. \]

Calibration of the accelerometer is instructive in itself. The position of the ball with the device flat on the table is the instrument's zero. It should be marked with an indelible marker. Rather than calculate the arc tangent for each scale marking

![Fig. 1. Student-built accelerometer. The accelerometer has been hot-melted to a plastic backing and calibrated to read from -4 to 4 miles/h/s.](image1)

![Fig. 2. Forces acting on the ball in the accelerometer tube.](image2)
to find an angle, the tangent can be constructed directly using two rulers, one laid flat on the tabletop (the “adjacent” side of the triangle) and the other standing upright (the “opposite” side of the triangle). The edge of the accelerometer is then tilted to form the hypotenuse of the triangle (Fig. 3). The accelerometer scale can either be in English or metric units, or both. By raising the end of the accelerometer incrementally, one can build the entire scale.

Since \( g = 21.9 \text{ miles/h/s} = 9.80 \text{ m/s/s} = 19.0 \text{ kt/s} \), the most natural scale division of 1 unit of acceleration occurs at these values:

- \( \tan \theta = 1/g = 1/22 \) for 1 mile/h/s, \( \tan \theta = 2/g = 2/22 \) for 2 miles/h/s…
- \( \tan \theta = 1/g = 1/10 \) for 1 m/s/s, \( \tan \theta = 2/g = 2/10 \) for 2 m/s/s…
- \( \tan \theta = 1/g = 1/19 \) for 1 kt/s, \( \tan \theta = 2/g = 2/19 \) for 2 kt/s…

Taking data with the accelerometer requires at least two people, or a video camera for later data transcription. One person monitors the clock and records that data called out by his/her partner on a predetermined time interval (ticks at every five seconds work well, but others may be more ambitious). Ultimately, these can be entered in a spreadsheet (see Fig. 4) for ease of calculation. For each interval, the change in velocity can be calculated by multiplying the accelerometer reading by the time interval. The running sum of the accelerations is the velocity as a function of time. A running sum of the velocity gives the displacement as a function of time. Any departure from the accelerometer reading or the calculated velocity returning to zero at the end of the run means that there was a problem. It could be that the vehicle was not level, the path was not level, or errors were made in taking data. Multiple runs add to the team’s ease in obtaining accurate and complete data. Running along the same path in the opposite direction helps to establish the route’s levelness. For advanced students, applying a correction proportionately to the data to make the average acceleration zero and/or the ending velocity zero can account for these problems.

Most runs between bus or subway stops involve a rather rapid acceleration at the start and a rather strong braking at the end. Subway drivers often test their brakes several seconds before they need to stop at a station. The accelerometer can be used to measure an airplane’s takeoff speed and length of the runway used. Unlike the driver of a bus or train, the pilot will release the plane’s brakes before starting down the runway only after running up the engines to takeoff power. Hence, the accelerometer reading will remain constant until the plane “rotates” and leaves the ground. Takeoff speed is then much easier to calculate. One simply multiplies the number of seconds traveling down the runway by the constant acceleration (e.g., a reading of 5 miles/h/s for 30 s gives a takeoff speed of 150 miles/h). The amount of runway used is the average speed times time elapsed (e.g., 0 to 150 miles/h gives an average speed of 75 miles/h for 1/120 h = 3300 ft). Pilots measure speed in knots (1 mile/h = 0.87 kt, 1 km/h = 0.54 kt). One can check results by asking a member of the crew to find the pilot’s actual takeoff speed and runway used.

Of course, an automobile on a flat road also works well, so long as there are no quick starts or stops that exceed the maximum acceleration measured by the accelerometer. However, the curiosity of the public about the activities of students tending to this odd device provides the added benefit of students both realizing that they have specialized knowledge and an occasion to practice their public speaking skills in describing how they are using physics to navigate in the same fashion as spacecraft and submarines.

References


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