

Inertial Navigation: A Bridge Between Kinematics and Calculus

Philip M. Sadler, Eliza N. Garfield, and Alex Tremblay, Harvard-Smithsonian Center for Astrophysics
Daniel J. Sadler, Rochester Institute of Technology

Those who come to Cambridge soon learn that the fastest route between Harvard and MIT is by the subway. For many students, this short ride is a quick and easy way to link physics and calculus. A simple, home-made accelerometer provides all the instrumentation necessary to produce accurate graphs of acceleration, velocity, and displacement position on this subterranean route. Each graph, in turn, reveals a story of the train's trip: speeding up and braking, varying track conditions, top speed, and the distance between subway stops deep beneath Massachusetts Avenue.

Using a horizontal accelerometer illustrates the fundamentals of inertial guidance, a reliable and accurate means to track the movements of airplanes, submarines, missiles, and spacecraft. Unlike GPS, there is no signal to be jammed nor interfered with by terrain or weather. It cannot be otherwise disturbed from afar and requires no radio frequency reception that can be compromised by intervening water or earth. No external reference is needed once initialized with a starting position and velocity. The key sensors in commercial and military "strap down" inertial navigation systems are three orthogonally oriented accelerometers. Gyroscopic stabilization provides an inertial reference frame. However, for simple systems in which one is interested in movement along a single dimension and on a level surface, one rudimentary accelerometer is sufficient.

While electronic accelerometers that record automatically are available for microcomputers and routinely are built into newer cell phones, it is far more instructive to construct one from commonly available materials. A bubble or ball in a curved tube is responsive to horizontal acceleration.¹ William U. Walton designed the first bubble-tube accelerometer and a means to calibrate it^{2,3} and incorporated it as a part of a popular introductory calculus curriculum in the 1970s.⁴ Others have "rediscovered" his invention over the ensuing 40 years, using it to measure angular acceleration,⁵ marketing versions for classroom use,⁶ or describing building versions for amusement park field trips.⁷ These simple accelerometers' tremendous potential to illustrate the principles of inertial navigation and their link to integral calculus merit their inclusion in any physics course.

Our accelerometer is built from a foot of clear flexible tubing (1/4-in i.d. works fine), two stoppers, and a single BB or stainless steel ball bearing, all commonly available items (Fig. 1). The tubing can be affixed to any convenient backing with tape or hot-melt glue and then filled with water. Any simple curve will work (we have had success with a 30° arc of a circle of 22-in [56-cm] diameter). The key to converting what looks

like a rudimentary bubble level into an accelerometer is calibration.

The physics is quite simple and similar to that of a pendulum except that the path of the mass is constrained to move along a circular arc by a plastic tube rather than a taut string (Fig. 2). The relevant forces acting on a bead anywhere along a curved tube are gravity and the force due to acceleration. When the forces normal to the arc add to zero, the bead is motionless. Its position along the arc indicates the value of the acceleration at that moment. Water in the tube dampens oscillations, so the position of the bead is easy to read.

$$F_1 = mg/\cos \theta, \quad (1)$$

$$F_2 = -ma/\sin \theta, \quad (2)$$

$$\Sigma \text{ forces} = F_1 + F_2 = mg/\cos \theta - ma/\sin \theta = 0 \quad (3)$$

$$a = g \sin \theta/\cos \theta = g \tan \theta. \quad (4)$$

Calibration of the accelerometer is instructive in itself. The position of the ball with the device flat on the table is the instrument's zero. It should be marked with an indelible marker. Rather than calculate the arc tangent for each scale marking

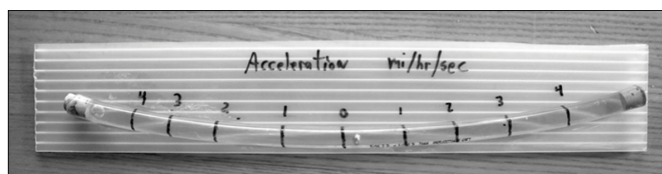


Fig. 1. Student-built accelerometer. The accelerometer has been hot-melted to a plastic backing and calibrated to read from -4 to 4 miles/h/s.

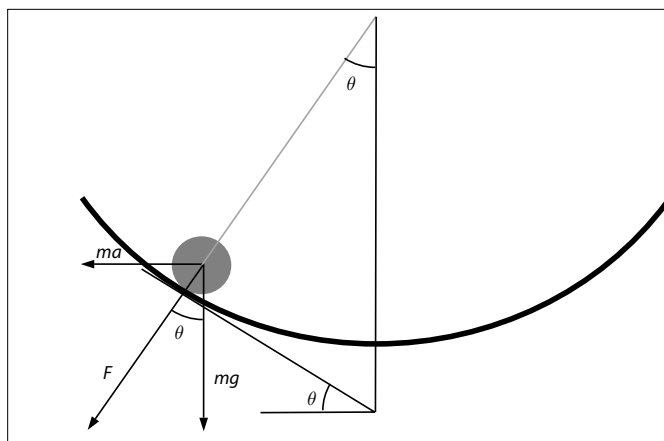


Fig. 2. Forces acting on the ball in the accelerometer tube.

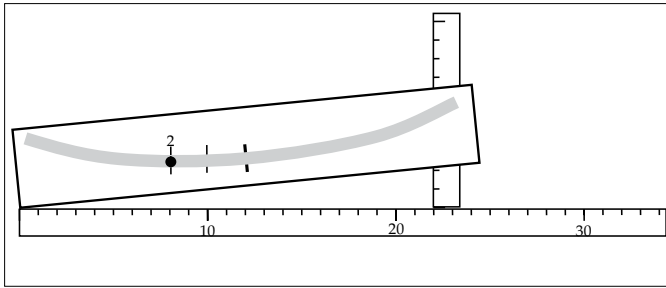


Fig. 3. Calibration of the accelerometer. The tangent of θ can be constructed directly without calculation by placing two metersticks at right angles. Tangent θ equal to $2/22$ or 2 miles/h/s is shown.

to find an angle, the tangent can be constructed directly using two rulers, one laid flat on the tabletop (the “adjacent” side of the triangle) and the other standing upright (the “opposite” side of the triangle). The edge of the accelerometer is then tilted to form the hypotenuse of the triangle (Fig. 3). The accelerometer scale can either be in English or metric units, or both. By raising the end of the accelerometer incrementally, one can build the entire scale.

Since $g = 21.9 \text{ miles/h/s} = 9.80 \text{ m/s/s} = 19.0 \text{ kt/s}$, the most natural scale division of 1 unit of acceleration occurs at these values:

$\tan \theta$ is: $1/g = 1/22$ for 1 mile/h/s, $2/g = 2/22$ for 2 miles/h/s ...
 $1/g = 1/10$ for 1 m/s/s, $2/g = 2/10$ for 2 m/s/s...
 $1/g = 1/19$ for 1 kt/s, $2/g = 2/19$ for 2 kt/s...

Taking data with the accelerometer requires at least two people, or a video camera for later data transcription. One person monitors the clock and records that data called out by his/her partner on a predetermined time interval (ticks at every five seconds work well, but others may be more ambitious). Ultimately, these can be entered in a spreadsheet (see Fig. 4) for ease of calculation. For each interval, the change in velocity can be calculated by multiplying the accelerometer reading by the time interval. The running sum of the accelerations is the velocity as a function of time. A running sum of the velocity gives the displacement as a function of time. Any departure from the accelerometer reading or the calculated velocity returning to zero at the end of the run means that there was a problem. It could be that the vehicle was not level, the path was not level, or errors were made in taking data. Multiple runs add to the team’s ease in obtaining accurate and complete data. Running along the same path in the opposite direction helps to establish the route’s levelness. For advanced students, applying a correction proportionately to the data to make the average acceleration zero and/or the ending velocity zero can account for these problems.

Most runs between bus or subway stops involve a rather rapid acceleration at the start and a rather strong braking at the end. Subway drivers often test their brakes several seconds before they need to stop at a station. The accelerometer can be used to measure an airplane’s takeoff speed and length of the runway used. Unlike the driver of a bus or train, the pilot

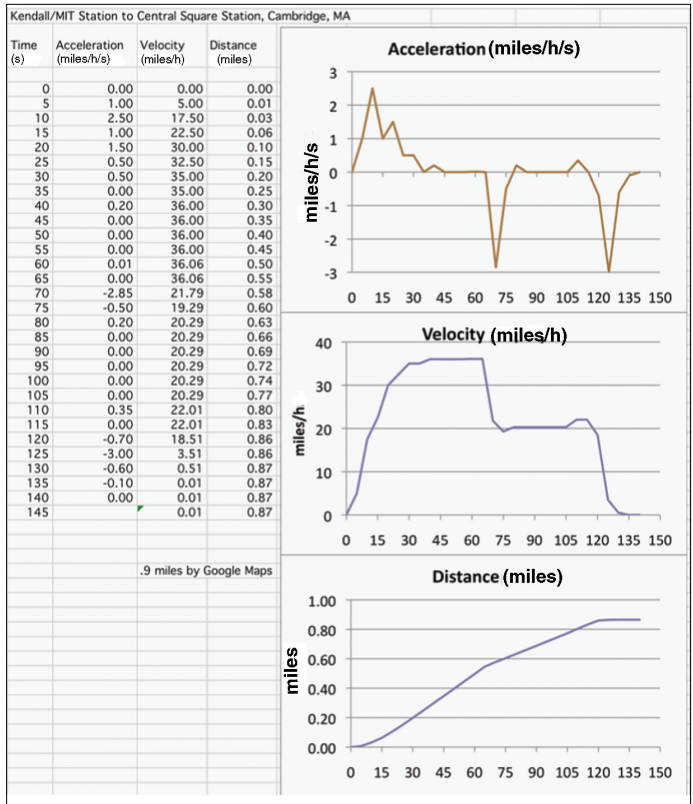


Fig. 4. Example of accelerometer data. Sample data show a 140-s run between subway stops. The top speed of the subway is 36 miles/h. The distance traveled is 0.87 miles.

will release the plane’s brakes before starting down the runway only after running up the engines to takeoff power. Hence, the accelerometer reading will remain constant until the plane “rotates” and leaves the ground. Takeoff speed is then much easier to calculate. One simply multiplies the number of seconds traveling down the runway by the constant acceleration (e.g., a reading of 5 miles/h/s for 30 s gives a takeoff speed of 150 miles/h). The amount of runway used is the average speed times time elapsed (e.g., 0 to 150 miles/h gives an average speed of 75 miles/h for $1/120 \text{ h} = 3300 \text{ ft}$). Pilots measure speed in knots (1 mile/h = 0.87 kt, 1 km/h = 0.54 kt). One can check results by asking a member of the crew to find the pilot’s actual takeoff speed and runway used.

Of course, an automobile on a flat road also works well, so long as there are no quick starts or stops that exceed the maximum acceleration measured by the accelerometer. However, the curiosity of the public about the activities of students tending to this odd device provides the added benefit of students both realizing that they have specialized knowledge and an occasion to practice their public speaking skills in describing how they are using physics to navigate in the same fashion as spacecraft and submarines.

References

- Hans Weltin, “Letters to the Editor: Accelerometer,” *Am. J. Phys.* **34**, 825–826 (Sept. 1966).

2. Harry M. Schey, Judah L. Schwartz, William U. Walton, and Jerrold R. Zacharias "A laboratory, computer, and calculus based course in mathematics," *Int. J. Math. Educ. Sci. Technol.* **1**, 115–13 (1970).
3. Solomon Garfunkel, "A laboratory and computer approach to calculus," *Am. Math. Mon.* **79**, 282–290 (March 1972).
4. William U. Walton, Harry M. Schey, David Christie, Solomon Garfunkel, and Philip M. Sadler, *Computer and Laboratory Calculus* (Education Development Center, Newton, MA, 1975).
5. C. A. Bunker, "Teaching elementary mechanics using a simple 'bubble tube' accelerometer," *Phys. Educ.* **21**, 272–276 (1986).
6. Science Kit, Tonawanda, NY; PASCO scientific, Roseville, CA.
7. Nathan A. Unterman, *Amusement Park Physics: A Teacher's Guide* (Walch Publishing, Portland, ME, 2001).

Philip M. Sadler, EdD, is F. W. Wright Senior Lecturer in Harvard's Department of Astronomy and Director of the Science Education Department at the Harvard Smithsonian Center for Astrophysics. Dr. Sadler has taught Harvard's courses for students preparing to be science teachers and for the next generation of science professors, as well as its oldest undergraduate course in science, *Celestial Navigation*. He invented the Starlab Portable Planetarium and led the development of the MicroObservatory Network of robotic telescopes. Materials and curricula developed by Dr. Sadler are used by an estimated 15 million students every year. He been honored with the American Astronomical Society Education Prize and the Journal of Research in Science Teaching Award.
psadler@cfa.harvard.edu

Eliza N. Garfield, EdD, is an education researcher at the Harvard Smithsonian Center for Astrophysics investigating the effectiveness of teacher professional development programs nationally. An expert on navigation, she holds a 1600 ton Oceans Masters license (ship captain) and has captained and navigated across the Atlantic several times. For four years, she captained the *Amistad*, a reproduction of the historic schooner that became the scene of the first successful shipboard slave revolt in the New World. She has also been a faculty captain at the Sea Education Association in Woods Hole, MA, and head teaching fellow for Astronomy 2 at Harvard, where she teaches celestial navigation.

Alex Tremblay was a teaching fellow for Astronomy 2, *Celestial Navigation* at Harvard. He is an avid pilot of single-engine airplanes. Currently, he works in the Washington, D.C., area as a software consultant.

Daniel J. Sadler is an undergraduate majoring in Mechanical Engineering Technology at Rochester Institute of Technology. He is interested in pursuing a career in design and testing military hardware and remote controlled vehicles when he graduates.



**I WANT YOU
FOR AN AAPT/PTRA
WORKSHOP LEADER**

If you are interested in applying to become an AAPT/Physics Teaching Resource Agent (PTRA) Professional Development provider, see the application at www.aapt.org/ptra.

If accepted, you will be invited to attend the AAPT/PTRA Leadership Institute occurring the week before the AAPT meeting in Philadelphia. All travel and on-site expenses are covered by PTRA funds.

Professional Development for Teachers
in Physics & Physical Science

www.aapt.org/ptra

